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COLLEGE ALGEBRA FOR FRESHMEN

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COLLEGE ALGEBRA FOR FRESHMEN

GORDON FULLER

Professor of Mathematics
Texas Technological College

SECOND EDITION

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PREFACE

Since many students who enter college have had only two semesters of high school algebra, and since most of the students who have studied algebra three or even four semesters need a substantial review before taking up advanced topics, the approach to college algebra for the vast majority of present-day freshmen must of necessity be elementary.

In the light of existing conditions, this text has been written for college freshmen as they are found to be, and not as one would wish them to be. The earlier chapters contain review material in which the treatment is sufficiently elementary and detailed for students who have had only one year of high school algebra. And the later chapters contain all the material which is usually taught in freshmen algebra courses. In this part, as in the introductory chapters, careful thought has been given to the reading ability of the student. The discussions and explanations are in clear and simple language, and a studied effort has been made to avoid a too brief or too mature treatment. Each new topic is amply illustrated with solved problems, these examples having been carefully planned to make clear the principles involved and to obviate troublesome points. Although manipulative processes and problem-solving techniques are emphasized, the chief aim is that of developing a real understanding of the subject. Thus a student should be better equipped for mastering the mathematics courses which follow and for attacking problems where algebra may be advantageously used.

As with the text proper, meticulous care has been used in making the exercises. The exercises begin with quite simple problems and increase in difficulty. Thus the less experienced students may gain confidence by doing the easier problems and be encouraged to further study; and the better prepared students will find sufficiently difficult problems to challenge and stimulate their best efforts. The lengths of the exercises are such that adequate drill should be furnished by assigning about one-third of the problems. Answers to the problems are included except for those whose numbers are multiples of three.

Although the text provides a course for students who need a detailed review of the fundamentals, it is also well suited to students who have a good foundation in high school mathematics. For such a group the review material could be covered rapidly or only in part. A quick review of the fundamental operations and of factoring may be accomplished through the supplementary problems at the end of each of these chapters.

The changes embodied in the new edition, both in material and exposition, are the result of the classroom experience of the author and the suggestions of other teachers who have used the first edition. The major changes comprise the following.

1. The problem sets are new, and virtually all the text proper has been rewritten.

2. The problems are more plentiful, making possible a longer use of the book without repetition in problem assignments.

3. Complex numbers are treated adequately in a new chapter. The first edition gave only a brief introduction to complex numbers.

4. In addition to the new chapter, several other topics are either new or have more detailed treatments. The maximum and minimum values of quadratic functions are discussed in the chapter on quadratic equations. The chapter on systems of quadratic equations contains a more careful consideration of the graphs of the conic sections and includes some additional methods of solving systems of second degree equations. Limits for the roots of polynomial equations and Descartes' rule of signs are included. Systems of linear equations with more equations than unknowns and systems of homogeneous linear equations are introduced.

5. The chapter on logarithms has been materially strengthened in exposition. One simple rule giving the relation of the decimal point of a number and the characteristic of its logarithm replaces the frequently used multiple rules. Logarithms to different bases are discussed.

G. F.

CONTENTS

1 REVIEW OF FUNDAMENTAL OPERATIONS 1

1. *Introduction.* **2.** *Real Numbers.* **3.** *Some Definitions.* **4.** *Addition.* **5.** *Subtraction.* **6.** *Grouping Symbols.* **7.** *Multiplication.* **8.** *Division.* **9.** *Operations with Zero.* **10.** *Special Products.* **11.** *The Product of the Sum and Difference of Two Numbers.* **12.** *The Square of a Binomial.* **13.** *The Square of a Multinomial.* **14.** *The Product of Two Binomials with Like Terms.* **15.** *The Cube of a Binomial.*

2 FACTORING 24

16. *Introduction.* **17.** *Rational Integral Expressions.* **18.** *Common Factors.* **19.** *The Difference of Two Squares.* **20.** *The Sum or Difference of Two Cubes.* **21.** *Trinomials.* **22.** *A Common Factor by Grouping.* **23.** *Multinomials Reducible to the Difference of Two Squares.* **24.** *Binomials of the Form* $x^n - y^n$ *and* $x^n + y^n$. **25.** *The Lowest Common Multiple.*

3 FRACTIONS 36

26. *Algebraic Fractions.* **27.** *Signs of Fractions.* **28.** *Reducing a Fraction to Lowest Terms.* **29.** *Multiplication and Division of Fractions.* **30.** *Addition and Subtraction of Fractions.* **31.** *Complex Fractions.*

4 EQUATIONS 50

32. *Equations.* **33.** *Operations on Equations.* **34.** *Solving Linear Equations in One Unknown.* **35.** *Equations with the Unknown in the Denominator.* **36.** *Stated Problems.*

5 FUNCTIONS AND GRAPHS 64

37. *Constants and Variables.* **38.** *Functions.* **39.** *Useful Notation for Functions.* **40.** *A Coordinate System.* **41.** *The Graph of a Function.* **42.** *The Graph of a Linear Function.*

6 SYSTEMS OF LINEAR EQUATIONS 73

43. *Linear Equations in Two Unknowns.* **44.** *Solution by Graphing.* **45.** *Solution by Algebraic Methods.* **46.** *Fractional Equations.* **47.** *Equations with Literal Coefficients.* **48.** *Linear Equations in Three Unknowns.* **49.** *Stated Problems Leading to Systems of Linear Equations.* **50.** *Determinants.* **51.** *Solution by Determinants.*

7 RATIO, PROPORTION, AND VARIATION 94

52. *Ratio.* **53.** *Proportion.* **54.** *Direct Variation.* **55.** *Inverse Variation.*

8 EXPONENTS AND RADICALS 103

56. *Laws of Exponents.* **57.** *Zero and Negative Integral Exponents.* **58.** *Roots of Numbers.* **59.** *Fractional Exponents.* **60.** *Equivalent Radical Forms.* **61.** *Addition and Subtraction of Radicals.* **62.** *Multiplication of Radicals.* **63.** *Division of Radicals.* **64.** *Imaginary Numbers.*

9 LOGARITHMS 124

65. *Logarithms.* **66.** *Laws of Logarithms.* **67.** *Common Logarithms.* **68.** *The Characteristic and Mantissa.* **69.** *Tables of Logarithms.* **70.** *Interpolation.* **71.** *Logarithms Used in Computations.* **72.** *Exponential Equations.* **73.** *Logarithms of a Number to Different Bases.*

10 QUADRATIC EQUATIONS 141

74. *Quadratic Equations in One Unknown.* **75.** *Solution by Factoring.* **76.** *Solution by Completing the Square.* **77.** *Solution by Formula.* **78.** *Equations in Quadratic Form.* **79.** *Equations Containing Radicals.* **80.** *The Nature of the Roots.* **81.** *The Sum and Product of the Roots.* **82.** *Graph of a Quadratic Function.* **83.** *Maximum and Minimum Values.* **84.** *Stated Problems.*

11 SYSTEMS INVOLVING QUADRATIC EQUATIONS 166

85. *Quadratic Equations in Two Unknowns.* **86.** *Graph of a Quadratic Equation.* **87.** *Solution by Graphing.* **88.** *Solution by Algebraic Methods.* **89.** *A System Without an xy and First Degree Terms.* **90.** *A System with a Linear and a Quadratic Equation.* **91.** *A System Without First Degree Terms.* **92.** *Other Methods of Solution.*

12 PROGRESSIONS 182

93. *Introduction.* **94.** *Arithmetic Progressions.* **95.** *The Sum of n Terms of an Arithmetic Progression.* **96.** *Geometric Progressions.* **97.** *The nth Term and the Sum of n Terms of a Geometric Progression.* **98.** *Infinite Geometric Progressions.*

13 INEQUALITIES 195

99. *Definition of Inequalities.* **100.** *Properties of Inequalities.* **101.** *Absolute and Conditional Inequalities.*

14 THE BINOMIAL THEOREM AND MATHEMATICAL INDUCTION . . 201

102. *Factorial n.* **103.** *The Binomial Theorem.* **104.** *The General Term of a Binomial Expansion.* **105.** *The Binomial Theorem for Fractional and Negative Exponents.* **106.** *Mathematical Induction.* **107.** *Proof of the Binomial Formula for Positive Integral Exponents.*

15 THEORY OF EQUATIONS 212

108. *Rational Integral Equations.* **109.** *The Remainder Theorem.* **110.** *The Factor Theorem and Its Converse.* **111.** *Synthetic Division.* **112.** *The Graph of a Polynomial.* **113.** *Theorems Concerning Roots.* **114.** *Imaginary and Quadratic Surd Roots.* **115.** *Coefficients in Terms of Roots.* **116.** *Limits for the Roots.* **117.** *Descartes' Rule of Signs.* **118.** *Rational Roots.* **119.** *A Method of Approximating Irrational Roots.* **120.** *Transformations of Equations.* **121.** *Horner's Method of Approximating Irrational Roots.*

16 COMPLEX NUMBERS 242

122. *The Fundamental Operations with Complex Numbers.* **123.** *Graphical Representation of Complex Numbers.* **124.** *The Polar Form of a Complex Number.* **125.** *Multiplication and Division of Numbers in Polar Form.* **126.** *De Moivre's Theorem.*

17 PARTIAL FRACTIONS 253

127. *Introduction.* **128.** *Case 1.* **129.** *Case 2.* **130.** *Case 3.* **131.** *Case 4.*

18 PERMUTATIONS AND COMBINATIONS 262

132. *Introduction.* **133.** *Permutations.* **134.** *Permutations of Things Not All Different.* **135.** *Circular Permutations.* **136.** *Combinations.*

19 PROBABILITY 273

137. *Introduction.* **138.** *Probability in Single Events.* **139.** *Probability in Multiple Events.* **140.** *Probabilities in Repeated Trials.* **141.** *Probability from Experiment.*

20 DETERMINANTS 283

142. *Introduction.* **143.** *Definition of a Determinant.* **144.** *Properties of Determinants.* **145.** *Expansion by Minors.* **146.** *Solution of Systems of Linear Equations by Determinants.* **147.** *Linear Systems with More Equations Than Unknowns.* **148.** *Systems of Homogeneous Linear Equations.*

TABLES 301

Table I. Powers and Roots.
Table II. Common Logarithms.
Table III. American Experience Mortality Table.
Table IV. Trigonometric Functions.

ANSWERS 307

INDEX 339

Chapter One

REVIEW OF FUNDAMENTAL OPERATIONS

1. Introduction. A student who plans to pursue courses in college mathematics should study algebra thoroughly and persistently. Algebra is basic to the courses which follow, and a good understanding of its principles augurs well for a continued efficient and pleasant study of mathematics.

Most students who study algebra in college have had a beginning course in high school algebra. But even with this previous experience a review is usually helpful. Consequently, the earlier chapters of this text treat quite elementary aspects of algebra.

2. Real Numbers. Arithmetic deals with positive numbers and zero. Early in the study of algebra, however, negative numbers are introduced. The positive and negative numbers and zero constitute the numbers of the **real number system.** They are called **real numbers** in distinction to the **imaginary** numbers, later to be defined.

A real number is either rational or irrational. If a number can be expressed as the quotient of two integers it is called a **rational number.** A number which is not expressible as the quotient of two integers is called an **irrational number.** The integers, or whole numbers, are rational, since each is equal to the quotient of itself and 1. Fractions like $\frac{1}{4}$, $-\frac{4}{7}$, and $\frac{8}{5}$ are rational. The numbers $\sqrt{2}$ and π are irrational. Neither is equal to the quotient of two integers.

The points of a straight line furnish a convenient and useful geometrical representation of the real numbers. The plan is as follows. Select any point of the line as the **origin.** Let this point represent the number 0. Then choose a unit length and measure the distances 1, 2, 3, and so on, to the right of the origin. The points thus determined represent the positive

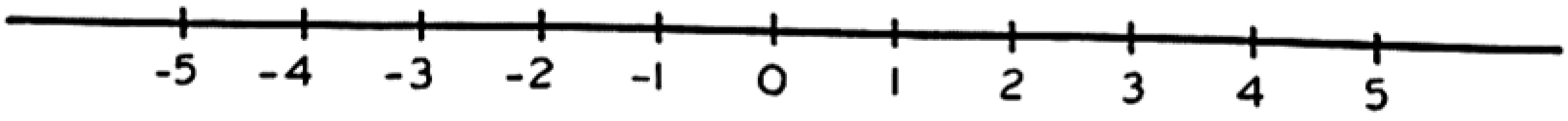


FIG. 1

integers. The points at distances 1, 2, 3, and so on, to the left of the origin represent the negative integers. A number between two consecutive integers has its point between the points of those integers. Thus the point for $\frac{1}{2}$ is midway between 0 and 1; the point corresponding to -1.25 is one fourth of the way from -1 to -2. The line of Figure 1, with its points representing numbers, is called a **number scale.**

A number M is greater than a number N if its point on the number scale is to the right of the point associated with N. This relation is also expressed by saying N is less than M.

The symbol $>$ is used to mean *is greater than* and the symbol $<$ means *is less than.* Employing these symbols, we may write

$$3 < 5 \quad \text{or} \quad 5 > 3;$$

$$-2 < 1 \quad \text{or} \quad 1 > -2;$$

$$-4 < -1 \quad \text{or} \quad -1 > -4.$$

The inequality symbol in every case points to the smaller of the two numbers.

3. Some Definitions. In addition to the specific numbers used in arithmetic, letters are used in algebra to stand for numbers. A letter may stand for any number or for certain restricted numbers. Numbers which are indicated by letters are called **literal numbers.** The operations of addition, subtraction, multiplication, and division are applied to them just as in arithmetic. The signs for those operations also apply. However, the multiplication sign is usually omitted in products involving literal numbers. Thus $3ab$ means the product of the numbers 3, a, and b.

Addition, subtraction, multiplication, and division are called the **fundamental operations** of algebra.

Each of two or more numbers which are multiplied together to form a product is called a **factor** of the product. In $7xy$, the numbers 7, x, and y are factors. The letters in a product, as x and y, are called **literal factors.** And usually the numerical factor, such as the 7, is called the **coefficient** of the other factors. But, more generally, any factor or factors may be regarded as the coefficient of the remaining factors. Thus in $7xy$, $7x$ is the coefficient of y; and $7y$ is the coefficient of x.

A single number or a grouping of numbers with some or all the fundamental operations indicated by the proper signs is called an **algebraic expression.** The plus (+) and minus (−) signs separate an expression into parts when there are more than one of these signs. Each of those parts with the sign preceding it is called a **term.** Frequently the first term of an expression has its sign omitted; in all such cases the plus sign is understood.

Illustrations:

$3a$ is an expression of one term.

$2x - 3y$ is an expression of two terms, the terms being $+2x$ and $-3y$.

$mn - pq + 7$ is an expression with the terms mn, $-pq$, and $+7$.

An expression of one term is called a **monomial**; an expression of two terms is called a **binomial**; and an expression of three terms is called a **trinomial.** The word **multinomial** is used to indicate an expression of two or more terms.

In algebraic expressions the terms which have the same literal factors are called **like terms.** In the expression $3xy - 3y - 2xy - 3x$ the terms $3xy$ and $-2xy$ are like terms.

4. Addition. The result of adding two or more numbers is called their **sum.** We assume that the operation of addition obeys the following laws:

1. THE COMMUTATIVE LAW. *The sum of two numbers is the same when the first is added to the second or when the second is added to the first.* Hence

$$4 + 5 = 5 + 4 \quad \text{and} \quad a + b = b + a.$$

2. THE ASSOCIATIVE LAW. *The sum of three or more numbers is the same in whatever manner they are grouped in adding.* Hence

$$\begin{aligned} 3 + 4 + 5 &= (3 + 4) + 5 = 7 + 5 \\ &= 3 + (4 + 5) = 3 + 9, \end{aligned}$$

and

$$\begin{aligned} a + b + c &= (a + b) + c \\ &= a + (b + c). \end{aligned}$$

In arithmetic positive numbers are added, but in algebra the operation of addition is performed on both positive and negative numbers. Addition in this wider system of numbers is called **algebraic addition.** Before stating the rules for algebraic addition, we shall define the absolute, or numerical, value of a number.

The **absolute,** or **numerical, value** of any positive number a is the number itself. The absolute value of a negative number $-a$ is a. Vertical bars are often used to denote the absolute value. Thus

$$|3| = 3, \quad |-3| = 3, \quad \text{and} \quad |0| = 0.$$

RULE 1. *The algebraic sum of two numbers of the same sign is the sum of their absolute values preceded by the common sign.*

RULE 2. *The algebraic sum of two numbers of different signs is the number obtained by subtracting the smaller absolute value from the larger absolute* *value and taking the sign of the number of greater absolute value. If the two numbers have different signs and the same absolute value, the sum is zero.*

RULE 3. *The algebraic sum of like terms is a term with the same literal factor or factors and whose coefficient is the algebraic sum of the coefficients of the terms.*

Henceforth, in speaking of addition, we shall mean algebraic addition as defined by these rules. The operation of Rule 3 is called **combining** or **collecting** like terms.

In the following illustrations of addition the quantities to be added are enclosed in parentheses.

Illustrations:

$$(11) + (-9) = 2; \qquad (-11) + (-9) = -20;$$

$$(-11) + (9) = -2; \qquad (a) + (-3a) + (4a) = 2a.$$

Addition may be interpreted graphically by use of the number scale (Fig. 1). To obtain the sum $(3) + (5)$, for example, we start at the origin and count 3 units to the right and then count 5 more units to the right. This brings us to 8, which is the sum. If we first count 5 units to the right and then 3 more units, we arrive again at 8. This illustrates the commutative law. To find the sum $(-5) + (3)$, we start at the origin and move 5 units to the left and then move 3 units to the right, stopping at -2. The same point would be reached, of course, by first moving 3 units to the right and then 5 units to the left.

Example. Add the expressions $3a - 7ab + 10b$, $-2a + 7ab + 12b$, and $ab - 14b - 7a$.

Solution. Write the expressions with like terms in columns. The order of the terms in the last expression is changed for this purpose.

$$\begin{array}{r}
3a - 7ab + 10b \\
-2a + 7ab + 12b \\
-7a + ab - 14b \\
\hline
-6a + ab + 8b
\end{array}$$

5. Subtraction. To subtract a number q from a number p means to find the number which added to q gives a sum equal to p. That is, $p - q = x$ if $q + x = p$. For example,

$$(13) - (5) = 8 \text{ because 8 added to 5 is 13.}$$

Similarly,

$$(13) - (-5) = 18 \text{ because 18 added to } -5 \text{ is 13.}$$

We note that in each of these examples the same result is obtained by changing the sign of the subtrahend (number to be subtracted) and then adding. That is,

$$(13) - (5) = (13) + (-5) = 8$$

and

$$(13) - (-5) = (13) + (5) = 18.$$

These results are in accordance with the rule for subtraction.

RULE. *To subtract one expression from another, change the signs of the terms in the expression to be subtracted and then add the two expressions.*

In the following illustrations the signs of the subtrahends are changed to emphasize the process. Usually it is preferable to make changes in signs mentally.

Illustrations:

$(9) - (4) = 9 - 4 = 5;$ $(9) - (-4) = 9 + 4 = 13;$

$(3x) - (4x) = 3x - 4x = -x;$ $(3x) - (-4x) = 3x + 4x = 7x.$

Example 1. From the sum of $8x + 7y - 4z$ and $-5x + 4y + 5z$ subtract $2x + 6y - 3z$.

Solution. We write the expressions with like terms in columns, and with the signs reversed in the last expression.

$$\begin{array}{r} 8x + 7y - 4z \\ -5x + 4y + 5z \\ -2x - 6y + 3z \\ \hline x + 5y + 4z \end{array}$$

The result could also be obtained by adding the first and second expressions and, from their sum, subtracting the last expression.

Example 2. Combine like terms of the expression

$$11a + 2ac + 6bc - 4 - 7a + 5ac - 3bc + 6 - a.$$

Solution. Rearrange by placing like terms together:

$$11a - 7a - a + 2ac + 5ac + 6bc - 3bc - 4 + 6 = 3a + 7ac + 3bc + 2.$$

The rearranging was done to make evident the collection process. This intermediate step may be omitted.

Exercise 1

Add as indicated:

1. $(14) + (6)$.
2. $(14) + (-6)$.
3. $(-14) + (6)$.
4. $(-14) + (-16)$.
5. $(7) + (-8)$.
6. $(-9) + (9)$.
7. $(-12) + (4)$.
8. $(-4) + (12)$.
9. $(9) + (-13)$.
10. $(2a) + (-8a)$.
11. $(-7a) + (7a)$.
12. $(-3a) + (-a)$.
13. $(xy) + (3xy)$.
14. $(-x) + (-4x)$.
15. $(3x) + (-7x)$.

16. $(18) + (-17) + (-10)$.
17. $(-21) + (24) + (-13)$.
18. $(-7) + (-9) + (-13)$.
19. $(-23) + (17) + (12)$.
20. $(5a) + (6a) + (-11a)$.
21. $(-2a) + (9a) + (-a)$.
22. $(14a) + (8a) + (-30a)$.
23. $(-8x) + (-4x) + (15x)$.
24. $(2ac) + (-6ac) + (9ac)$.
25. $(5mn) + (-mn) + (7mn)$.
26. $(mn) + (-mn) + (6mn)$.
27. $(-3y) + (-7y) + (-6y)$.

Perform the subtractions:

28. $(12) - (4)$.
29. $(16) - (-7)$.
30. $(-5) - (12)$.
31. $(-7) - (4)$.
32. $(-8) - (-5)$.
33. $(17) - (14)$.
34. $(6) - (-6)$.
35. $(6) - (6)$.
36. $(-20) - (13)$.
37. $(3x) - (2x)$.
38. $(5x) - (-3x)$.
39. $(-x) - (-7x)$.
40. $(mn) - (-4mn)$.
41. $(-3mn) - (7mn)$.
42. $(2mn) - (4mn)$.

43. $(13) - (-17) - (10)$.
44. $(-9) - (-6) - (7)$.
45. $(-7) - (1) - (-6)$.
46. $(8) - (-4) - (12)$.
47. $(4x) - (3x) - (7x)$.
48. $(5x) - (-6x) - (-x)$.
49. $(6c) - (12c) - (-8c)$.
50. $(-4c) - (-c) - (2c)$.
51. $(-8a) - (-3a) - (-2a)$.
52. $(5a) - (6a) - (7a)$.

Add the two expressions in each problem **53–69.** Next subtract the second expression from the first.

53. $\begin{array}{r} x + 2y - 8 \\ 3x - 4y + 9 \\ \hline \end{array}$

54. $\begin{array}{r} 2x + 10y + 16 \\ 7x - 8y + 10 \\ \hline \end{array}$

55. $\begin{array}{r} 2a - 3b - c \\ a + 5b - 3c \\ \hline \end{array}$

56. $\begin{array}{r} 4a - 6b - 7c \\ -a + 5b + 3c \\ \hline \end{array}$

57. $\begin{array}{r} 5a + 9b - 10c \\ -6a \qquad + 7c \\ \hline \end{array}$

58. $\begin{array}{l} 3x - 3y - z \\ 2x + 5y \\ \hline \end{array}$

59. $\begin{array}{l} 7x + 7y \\ 3x - 9y + 6z \\ \hline \end{array}$

60. $\begin{array}{r} 3m - 10n - 13 \\ -m + 13n + 23 \\ \hline \end{array}$

61. $\begin{array}{r} 11m - 7n + 13 \\ 3m - 8n - 21 \\ \hline \end{array}$

62. $10a - 17b + 24c$; $13a + 14b - 16c$.
63. $3a + 4b - 5c + 3$; $6a + 7b - 8c - 5$.
64. $15b + 16c - 17a$; $12a - 13b + 16c$.
65. $16x - 11y - 14z$; $10z + 17x - 11y$.
66. $4x - 7y + 6z + 5$; $6x + 3y - 5z + 7$.

67. $5x + 6xy + 3yz;\ 3x + 7xy - 6yz.$
68. $-2a + 7b + 3c - 4;\ 5a + 6b - 4c - 11.$
69. $4a + 16b + 17c;\ -10a + 7 - 11c + 14b.$

Add the three expressions in each problem **70–77.** Next subtract the third expression from the sum of the first and second.

70. $2x + 7y - 3;\ 5x - 6y - 5;\ 4x - 5y + 7.$
71. $a - 3b + 4c;\ 6a + 7b - 3c;\ -a + 4b + 7c.$
72. $3a - 4b - 7;\ -6a + 7b + 2;\ 5a - b - 3.$
73. $7x - 5y - 4;\ 5x + 4y + 6;\ 10x - 11y - 13.$
74. $2x - y - 25;\ 16 - 4y - 13x;\ 5y - 6x - 30.$
75. $-9x + 7y + 19;\ 7x - 13y - 15;\ 16x - 14 + 7x.$
76. $x - 14y - 16z;\ -4x + 18y + 13z;\ 15x - 19y + 14z.$
77. $7a - 3b + 11c;\ -14a + 10b + 10c;\ 8a + 8b + 13c.$

78. From the sum of $2x - 4y + 2z$ and $3x + 5y - 8z$ subtract the sum of $x - y + 7z$ and $6x + 4y - 9z$.

79. From the sum of $5a - 7b + 6c$ and $4a - 4b - 5c$ subtract the sum of $3a - 9b - 11c$ and $-a + 2b - 3c$.

80. Add $5x - 6y + 7z$, $6x + 7y + 10z$, and $8x + 4y - z$. Subtract the result from $10x - 4y$.

81. From $11x - 13y + 17z$ subtract the sum of $2x - y - z$, $3x + 2y + 3z$, and $x - y - 6z$.

Combine like terms:

82. $3x + 7y - 3z + 6x - 8y - 7z + 5 - 1.$
83. $5 - 2a + 4b - 6 - 7c + 8b + 10a - 4.$
84. $3mn + 2m - 4n + 3m - 6n + 7mn + 7 + n.$
85. $4x - 7 + 5xy + 6y + 8 - 2xy + 2x + y - 5.$
86. $7y + 3c - 4a - 11 + 7c - 9y + 6 - a + 6.$
87. $2x + 3y + 6 - 5 + z - 2y - 4x + 6x - 5 + 4y.$
88. $3ab - 2ac + 4bc - 5 - 6bc - 8ab + 11ac.$
89. $ac - 2a + 3c + b - 4 + 2b + 6ac + 7a - 3c.$
90. $4 + m + 2n - 5m + 6 - 3mn + mn - 3n + 4.$
91. $xy - 2xz + 4yz - 5 + 5yz - 2xy + 2xz - 9yz + 4.$
92. $m + a - 4 + c - 5c + 6m + 4a + 7c - 2 + 3m.$
93. $9xy + 3x + 4a - 5ax + 10a - 7x + 3xy + 6ax.$

6. Grouping Symbols. It is often desirable to think of an expression of two or more terms as a single quantity. An expression may be thus singled out or set apart by enclosing it in grouping symbols. The grouping symbols generally used are **parentheses** ( ), **brackets** [ ], and **braces** { }.

We may indicate that two expressions are to be multiplied by enclosing each expression in grouping symbols and writing them side by side. For example, $(2x - 3y)(x + 4y)$ means to multiply $x + 4y$ by $2x - 3y$. Similarly, $(2x - 3y) - (x + 4y)$ means that $x + 4y$ is to be subtracted from $2x - 3y$.

In performing algebraic operations it is often necessary to remove grouping symbols. For example, to carry out the operation indicated by $(3x - 5y) - (2x - 3y)$, we first remove the parentheses from each expression. Since no sign precedes the parentheses enclosing the first expression, a plus sign is understood. Hence we omit the parentheses and write the enclosed expression as it is. The minus sign preceding the second expression means to subtract the enclosed expression. Consequently, employing the rule for subtracting, we reverse the signs of the enclosed expression. Thus we have

$$(3x - 5y) - (2x - 3y) = 3x - 5y - 2x + 3y = x - 2y.$$

For inserting or removing parentheses (or other grouping symbols), we have the following rules:

RULE 1. *Parentheses preceded by a plus sign may be placed about, or removed from, an expression if no change is made in the signs of the expression.*

RULE 2. *Parentheses preceded by a minus sign may be placed about, or removed from, an expression if the sign of each of the terms is changed.*

Frequently one or more sets of grouping symbols are contained within another. To remove such multiple sets, it is better to begin with the innermost symbols.

Illustrations:

$$\begin{aligned} -[x - (-2x + 3) + 5 - 4x] &= -[x + 2x - 3 + 5 - 4x] \\ &= -[-x + 2] = x - 2. \end{aligned}$$

$$\begin{aligned} \{6x - [3x - (x - y) - 4y] + y\} &= \{6x - [3x - x + y - 4y] + y\} \\ &= \{6x - [2x - 3y] + y\} \\ &= \{6x - 2x + 3y + y\} \\ &= 4x + 4y. \end{aligned}$$

Exercise 2

Remove the symbols of grouping and simplify by combining like terms:

1. $3a - (b + c) + (a + b - c)$.

2. $4x + (y - 3) - (2x + 1)$.

3. $(3x - 2y) - (x + 4y - 2)$.

4. $(x - y) - (2x - 3y) - (-x + y)$.

5. $a - (b + c - 4) + (a + c + 2)$.

6. $1 - [a - 2b - (3 - a) + 3]$.

7. $8a - [(a + 3b) - (3a - b)]$.

8. $-[x + (3 - x) - (4 + 3x)]$.

9. $(3x - 2y + 4) - [x - (x + y) - 3y + 2]$.
10. $-\{5a - b - [3b - (c - 2b + a) - 4a] + c\}$.
11. $3xy - \{-(2xy + 4x) + [3y - (x - xy + x + 2xy)]\}$.
12. $-\{a - 2ab + b - [3a + 5ab + 6b - (a - b) + 5]\}$.
13. $5 + \{3x + 2y - [4 - x - 10y + (2y - 7x + 2) - 3] + 8x - y\}$.
14. $-\{4 - (7x - 8y + 3) - [(3x - 2y + 6) - (x + 2y) - 5] + (-6x + y)\}$.
15. $7a - \{(-4ab - b + 10) - [-(4a - 6ab - 5) + (-7 + b + 5)]\}$.

7. Multiplication. The result of multiplying two or more numbers is called their **product.** The sign × or · placed between numbers indicates multiplication. Multiplication is also indicated by writing letters with no sign between them. Thus the product of a and b may be written as $a \times b$, $a \cdot b$, or simply as ab.

We assume that the operation of multiplication obeys the following laws:

1. THE COMMUTATIVE LAW. *The product of two numbers is the same when the first number is multiplied by the second or when the second number is multiplied by the first.* Hence

$$3 \times 5 = 5 \times 3 \quad \text{and} \quad ab = ba.$$

2. THE ASSOCIATIVE LAW. *The product of three or more numbers is the same in whatever manner they are grouped in multiplying.* Hence

$$\begin{aligned} 3\cdot 4\cdot 6 &= (3\cdot 4)\cdot 6 = 12\cdot 6 = 72 \\ &= 3\cdot(4\cdot 6) = 3\cdot 24 = 72, \end{aligned}$$

and

$$abc = (ab)c = a(bc).$$

3. THE DISTRIBUTIVE LAW. *The product of a number and the sum of numbers is the same as the sum of the products of the first number by each of the numbers forming the sum.* Hence

$$\begin{aligned} 3\cdot(4 + 6) &= 3\cdot 10 = 30 \\ &= 3\cdot 4 + 3\cdot 6 = 12 + 18 = 30, \end{aligned}$$

and

$$a(b + c) = ab + ac.$$

In multiplying two or more numbers it is necessary to know what sign to give the product. Rules for this purpose may be formulated if we notice that multiplication is essentially addition. Thus $(3)(5)$ means $5 + 5 + 5 = 15$. Here both factors are positive and the product is positive. The product $(3)(-5)$ may be interpreted as $-5 - 5 - 5 = -15$. Since multiplication is commutative, then $(-5)(3) = -15$. Thus we obtain a negative number as the product of two numbers of unlike signs. Finally, we interpret the product $(-5)(-3)$ as the negative of the product $(5)(-3)$. This makes $(-5)(-3) = -(5)(-3) = -(-15) = 15$, a positive result.

We adopt rules for multiplication which are in harmony with the results of the preceding examples.

Rule 1. *The product of two numbers of like signs is positive. The product of two numbers of unlike signs is negative.*

Corollary. *The product of an even number of negative factors is positive. The product of an odd number of negative factors is negative.*

Illustrations:

$$(-2)(-3) = 6; \quad (-2)(3) = -6;$$

$$(-2)(-3)(4) = (6)(4) = 24;$$

$$(-2)(3)(4) = (-6)(4) = -24;$$

$$(-2)(-3)(-4) = (6)(-4) = -24.$$

Products often have the same factor occurring two, three, or more times. This may be represented by writing the factor the proper number of times. But it is more convenient to use a shorthand notation. For this purpose we have the following definitions:

$$a \cdot a = a^2 \quad \text{(read "a square")},$$

$$a \cdot a \cdot a = a^3 \quad \text{(read "a cube")},$$

$$a \cdot a \cdot a \cdot a = a^4 \quad \text{(read "a fourth")}.$$

The numbers 2, 3, and 4 in a^2, a^3, and a^4 are called **exponents.** They tell how many times the number a is to occur as a factor. The quantities a^2, a^3, and a^4 are called the second, third, and fourth **powers** of a. In general, a^m (m a positive integer) means that a is to be used m times as a factor. The product a^m is called the **mth power of a.** The number a is called the base and m is called the exponent. The first power of a number is usually expressed without an exponent. Thus $a = a^1$.

Next let us note how we can find the product of two powers of the same base, that is, products like a^2a^3 and $2^3 \cdot 2^4$. By definition $a^2 = a \cdot a$ and $a^3 = a \cdot a \cdot a$. Hence $a^2a^3 = a \cdot a \cdot a \cdot a \cdot a = a^5$. Similarly, $2^3 \cdot 2^4$ means a product in which 2 is a factor $3 + 4$ times. We, therefore, have

$$a^2a^3 = a^{2+3} = a^5 \quad \text{and} \quad 2^3 \cdot 2^4 = 2^{3+4} = 2^7.$$

Now consider the more general product $a^m a^n$, where m and n are any positive integers. The quantity a^m means the product of m factors, each of which is a, and a^n means a is a factor n times. In all, a occurs as a factor $m + n$ times, and consequently

$$a^m a^n = a^{m+n}.$$

The product of two powers of the same base is a number with the same base and an exponent equal to the sum of the two exponents.

Illustrations:

$$(2a^3b^2)(3ab^4) = 6a^4b^6;$$

$$(-4x^2y^2)(xy^2)(5xy) = -20x^4y^5.$$

The following examples show how to find products involving multinomials.

Example 1. Multiply $x^2 - 3xy - 2y^2$ by $2y^3$.
Solution.

$$\begin{aligned} 2y^3(x^2 - 3xy - 2y^2) &= 2y^3(x^2) + 2y^3(-3xy) + 2y^3(-2y^2) \\ &= 2x^2y^3 - 6xy^4 - 4y^5. \end{aligned}$$

This result, obtained by multiplying each term of the trinomial by $2y^3$, is an application of the distributive law for multiplication. The intermediate step is written to emphasize the process. The usual procedure is to write only the final result.

Example 2. Multiply $x^2 - 3x^3 + 5x - 6$ by $2x - 3$.
Solution. Write the terms in order of descending powers of x.

$$\begin{array}{rrrrrl} & -3x^3 & +\ x^2 & +\ 5x & -\ 6 & \\ & & & 2x & -\ 3 & \\ \hline -6x^4 & +\ 2x^3 & +\ 10x^2 & -\ 12x & & \text{(Each term multiplied by } 2x\text{)} \\ & 9x^3 & -\ 3x^2 & -\ 15x & +\ 18 & \text{(Each term multiplied by } -3\text{)} \\ \hline -6x^4 & +\ 11x^3 & +\ 7x^2 & -\ 27x & +\ 18 & \text{(Like terms combined)} \end{array}$$

This scheme for multiplying is embodied in the following rule. The rule may be fully justified by the commutative, associative, and distributive laws of multiplication.

RULE 2. *The product of two multinomials is the sum of all the results obtained by multiplying all the terms of one multinomial by each term of the other.*

Exercise 3

Multiply as indicated:

1. $(5)(-4)(-2)$.
2. $(3)(-1)(7)$.
3. $(-2)(-3)(-7)$.
4. $(3ab)(2a)$.
5. $(7a)(-4b)$.
6. $(5xy)(2xy)$.
7. $(-2a^2)(-3a)$.
8. $(4mn)(-3m^2n)$.
9. $(-6a^2b)(3ab^2)$.
10. $(xy)(-2y)(3x)$.
11. $(-2a)(3b)(-4ab)$.
12. $(3xy^2)(5x^2y)(xy)$.
13. $(-ab)(-a^2b)(-b^2)$.
14. $(-xy)(2y)(-5x)$.
15. $(xy^2)(-x^3y)(2xy)$.
16. $4x(x - 3y)$.
17. $-2a(3a - 2b)$.
18. $5m(4 - 2m)$.
19. $-7a^2(3a - 4)$.
20. $xy(x^2 - y^2)$.
21. $3x^2y(2xy^2 + y)$.
22. $(x + 3y)(x - 3y)$.
23. $(2a - b)(2a + b)$.
24. $(3x + 2y)(3x - 2y)$.
25. $(x - 2)(x + 1)$.
26. $(a + 3b)(2a - b)$.
27. $(2x + 3)(3x - 2)$.
28. $(2m - 3n)(m - 4n)$.
29. $(4m + 5)(3m - 6)$.
30. $(1 - 2x)(3 + 4x)$.
31. $3x^2(4x^2 - 7x + 2)$.
32. $2ab(a^2 - 4ab + 3ab^2)$.
33. $2x^2(-4y + 3xy + x)$.
34. $-3xy^2(2x + 3xy - 4y + 1)$.
35. $-4mn(3 - 5m + 6mn + 3n)$.
36. $5a^2(4a^2 - 5ab + b^2)$.
37. $7xy(2y - 4xy + y^2 + 3)$.
38. $-x^2y^2(x^3 - 2xy + 3xy^2)$.
39. $(x - 1)(x^2 + x + 1)$.
40. $(a + b)(a^2 - ab + b^2)$.
41. $(2a + 3)(4a^2 - 6a + 9)$.
42. $(m - 5n)(m^2 + 5mn + 25n^2)$.
43. $(y + 2x)(y^2 + xy - x^2)$.
44. $(c - 4a)(2c^2 - 5ac + a^2)$.
45. $(a - 3b)(3a^2 + 6ab + 4b^2)$.
46. $(3a + 2c)(a^2 + 7ac - 8c^2)$.
47. $(b^2 + 2)(2b^2 - 3b + 5)$.
48. $(x^2 - 7)(3x^2 + 5x + 6)$.
49. $(a + b + 1)(a - b + 1)$.
50. $(2x - y + 3)(2x + y + 3)$.
51. $(m + n + 1)(m + n - 2)$.
52. $(x^2 - 2x + 3)(x^2 + 3x - 2)$.
53. $(a^2 + 3a + 4)(a^2 - 2a - 3)$.
54. $(c^2 - 3c + 2)(c^2 + c - 4)$.
55. $(2x^2 - x - 7)(3x^2 + x + 4)$.
56. $(m^2 - 4m - 3)(3m^2 + m - 4)$.
57. $(x - 1)(x - 2)(x - 3)$.
58. $(a + 2)(a - 1)(a + 3)$.
59. $(2x + 1)(x - 3)(x + 5)$.
60. $(3a + 2)(a - 2)(2a + 1)$.
61. $(3 - m)(2 - 3m)(1 - m)$.
62. $(4 - x)(4 - 2x)(4 + x)$.

8. Division. To divide a by b means to find a number which multiplied by b gives a. That is, $a \div b = x$ if $bx = a$. The number a is called the dividend, b the **divisor,** and x the **quotient.** The signs — and / are also used to indicate division. Thus, a divided by b may be written as $a \div b$, $\frac{a}{b}$, or a/b.

Illustrations:

$$\frac{12}{4} = 3; \quad \frac{-12}{-4} = 3; \quad \frac{12}{-4} = -3.$$

In each case the product of the quotient and divisor is equal to the dividend. Division, being defined in terms of multiplication, has its rule of signs in conformity with those for multiplication.

RULE. *The quotient of two numbers of like signs is positive. The quotient of two numbers of unlike signs is negative.*

If m and n are positive integers and $m > n$, then

$$\frac{y^m}{y^n} = y^{m-n}.$$

This result is correct because the quotient times the divisor is equal to the dividend. That is, $y^{m-n} \cdot y^n = y^{m-n+n} = y^m$.

The quotient of a power of a base and a smaller power of the same base is a number with the same base and an exponent equal to the difference of the two exponents.

Illustrations:

$$\frac{y^5}{y^2} = y^{5-2} = y^3; \quad \frac{15x^6}{-5x^4} = -3x^{6-4} = -3x^2$$

Example 1. Divide $3x^2y - 6xy^2 + 12x$ by $3x$.
Solution.

$$\frac{3x^2y - 6xy^2 + 12x}{3x} = xy - 2y^2 + 4.$$

Each term of the multinomial is divided by the monomial and the result expressed as an algebraic sum.

Example 2. Divide $3x - 6x^2 + 18$ by $2x + 3$.
Solution. Arrange the dividend and divisor in order of descending powers of x, and place the work thus:

$$\begin{array}{llrl}
 & & -3x + 6 & \text{(Quotient)} \\ \cline{3-3}
\text{(Divisor)} & 2x + 3\big) & -6x^2 + 3x + 18 & \text{(Dividend)} \\
 & & -6x^2 - 9x \phantom{{}+18} & \\ \cline{3-3}
 & & 12x + 18 & \\
 & & 12x + 18 & \\ \cline{3-3}
\end{array}$$

Hence we have

$$\frac{-6x^2 + 3x + 18}{2x + 3} = -3x + 6.$$

Explanation.

1. Divide the first term of the dividend by the first term of the divisor $(-6x^2 \div 2x = -3x)$ to obtain the first term of the quotient.

2. Then multiply the entire divisor by $-3x$ and write the product $(-6x^2 - 9x)$ under like terms of the dividend.

3. Subtract and obtain $12x + 18$, a new dividend.

4. Next divide the first term of the new dividend by $2x$ and obtain 6, the second term of the quotient.

5. Multiply the divisor by 6 and place the result under the new dividend. Subtraction now yields zero.

Example 3. Divide $6a^4 - 41a^2 + 3a + 6$ by $2a^2 - 4a - 3$.

Solution.

$$\begin{array}{rllllll}
 & 3a^2 & + 6a & - 4 & & & \text{(Quotient)}\\ \cline{2-6}
2a^2 - 4a - 3) & 6a^4 & & - 41a^2 & + 3a & + 6 & \text{(Dividend)}\\
 & 6a^4 & - 12a^3 & - 9a^2 & & & \\ \cline{2-4}
 & & 12a^3 & - 32a^2 & + 3a & & \\
 & & 12a^3 & - 24a^2 & - 18a & & \\ \cline{3-5}
 & & & - 8a^2 & + 21a & + 6 & \\
 & & & - 8a^2 & + 16a & + 12 & \\ \cline{4-6}
 & & & & 5a & - 6 & \text{(Remainder)}
\end{array}$$

Hence

$$\frac{6a^4 - 41a^2 + 3a + 6}{2a^2 - 4a - 3} = 3a^2 + 6a - 4 + \frac{5a - 6}{2a^2 - 4a - 3}.$$

The remainder in this division is not zero. Consequently we express the result in the form used in arithmetic when the division is inexact.

A problem in division can be checked by using the relation

$$\text{Dividend} = \text{quotient} \times \text{divisor} + \text{remainder}.$$

9. Operations with Zero. The number 0 plays a unique and important role in mathematics. Its properties in the fundamental operations differ from those of any other number. Its use for these operations conforms to the following rules and illustrations.

If zero is added to or subtracted from any number, the result is equal to the original number. As examples,

$$2 + 0 = 2; \quad 2 - 0 = 2; \quad -4 + 0 = -4.$$

The product of zero and any number is zero. Thus

$$6 \cdot 0 = 0; \quad -9 \cdot 0 = 0; \quad 0 \cdot 0 = 0.$$

If zero is divided by any number different from zero, the quotient is zero. Hence

$$\frac{0}{4} = 0; \quad \frac{0}{-6} = 0; \quad \frac{0}{a} = 0.$$

These results follow immediately from the definition of division. In each the quotient times the divisor gives zero, which is the dividend.

Zero is never to be used as a divisor.

The reason for rejecting zero as a divisor becomes evident when we try to use it for this purpose. To divide a number k (different from zero) by zero would mean to find a number which multiplied by zero gives k. This is impossible, of course, since the product of zero and any number is zero. This might suggest that zero may be used as a divisor if the dividend is also zero. In this case, however, the quotient could be any number k because $k \cdot 0 = 0$. Hence division by zero is never allowed.

Exercise 4

Divide as indicated:

1. x^2/x^2.
2. $-y^3/y^3$.
3. $3a/3a$.
4. b^4/b^2.
5. a^8/a^5.
6. x^6/x^2.
7. $-y^4/y^3$.
8. acz/ac.
9. $10ab/-5ab$.
10. $12a^4/6a$.
11. $-9c^3d^4/3c^3d^3$.
12. $21x^2y^2/21xy^2$.
13. $34a^3b^2/17a^2b$.
14. $40x^2y^3/-8xy^2$.
15. $-18m^2n^2/-3mn$.
16. $\dfrac{6x^2 - 3x}{3x}$.
17. $\dfrac{7x^2 - 21x^3}{7x^2}$.
18. $\dfrac{15a^2b^3 + 9ab}{3ab}$.
19. $\dfrac{9x^2 - 6x - 3}{3}$.
20. $\dfrac{m^2n - mn^2 + mn}{mn}$.
21. $\dfrac{7x^2y + xy + xy^2}{xy}$.
22. $\dfrac{25x^3 - 5x^2 - 10x}{5x}$.
23. $\dfrac{8a^3 - 4a^2 - 8a}{-4a}$.
24. $\dfrac{2x^4 - 6x^3 - 4x^2}{-2x^2}$.

Perform the division indicated in each of the following problems. Check each result by finding the product of the divisor and quotient and adding the remainder.

25. $\dfrac{x^2 - 5x + 4}{x - 1}$.
26. $\dfrac{x^2 - 7x - 16}{x + 2}$.
27. $\dfrac{3x^2 + x - 24}{x + 3}$.
28. $\dfrac{6x^2 + 5x - 4}{2x - 1}$.
29. $\dfrac{12x^2 - 7x - 10}{3x - 4}$.
30. $\dfrac{12x^2 + 3x + 10}{4x + 5}$.
31. $\dfrac{8 - 6x + x^3}{x + 3}$.
32. $\dfrac{x^3 - 2x^2 - 32}{x - 4}$.
33. $\dfrac{2x^3 - 20x - 5}{2x + 6}$.
34. $\dfrac{a^3 - b^3}{a - b}$.
35. $\dfrac{a^3 + b^3}{a + b}$.
36. $\dfrac{a^4 - b^4}{a - b}$.
37. $\dfrac{x^3 + 10x^2 + 12x + 27}{x^2 + x + 3}$.
38. $\dfrac{4 - 7x + 10x^2 + 12x^3}{2x^2 + x + 1}$.

39. $\dfrac{x^4 - 2x^3 - 7x^2 + x - 10}{x - 4}$.
40. $\dfrac{4x^4 - 8x^2 - 2x^3 + 1 + 3x}{2x - 3}$.
41. $\dfrac{2x^4 - 5x^3 + 11x - 2x^2 - 6}{x^2 - 3x + 2}$.
42. $\dfrac{4x^4 - 2x^3 + 6x^2 + x + 3}{2x^2 + 1 + x}$.

43. $(4m^3 + 10m^2n - 5mn^2 + 3n^3) \div (m + 3n)$.
44. $(3m^3 - 2m^2n - 3mn^2 + 2n^3) \div (3m - 2n)$.
45. $(2m^4 - 3m^3n + 3m^2n^2 + mn^3 - n^4) \div (2m - n)$.
46. $(x^5 - 5x^4 + 7x^3 + x^2 - 8x + 7) \div (x - 2)$.
47. $(6 + 10x + 3x^2 + x^3 + x^4 + x^5) \div (x^2 + x + 2)$.
48. $(2x^5 + 5x^2 + 1 - 5x^4 - 4x^3 - 5x) \div (2x - 1)$.
49. $(2x^5 - 5x^4 + 6x^3 + 4x^2 - 11x + 4) \div (2x^2 - 3x + 1)$.

10. Special Products. Certain kinds of algebraic expressions occur over and over in multiplications. Consequently it is worthwhile to learn to write their products speedily. We shall derive formulas, or rules, for multiplying which greatly condense the procedure used in Section 7. The rules should be memorized and applied until proficiency is acquired.

11. The Product of the Sum and Difference of Two Numbers. Indicating one number by x and the other by y, we write the product of their sum and difference as

$$(x + y)(x - y) = x^2 - y^2.$$

This is a simple formula for the product of the sum and difference of two numbers. We state the formula in words.

The product of the sum and difference of two numbers is equal to the square of the first minus the square of the other.

Example 1. Perform the multiplication $(3x + 4y)(3x - 4y)$.
Solution. $(3x + 4y)(3x - 4y) = (3x)^2 - (4y)^2 = 9x^2 - 16y^2$.

Example 2. Perform the multiplication $(42)(38)$.
Solution. $(42)(38) = (40 + 2)(40 - 2) = 1600 - 4 = 1596$.

Exercise 5

Apply the formula and write the products:

1. $(m + 4)(m - 4)$.
2. $(2x + 3)(2x - 3)$.
3. $(4x + 3)(4x - 3)$.
4. $(5 + m)(5 - m)$.
5. $(2m + n)(2m - n)$.
6. $(5a + 2b)(5a - 2b)$.
7. $(4 - 3x)(4 + 3x)$.
8. $(x^2 + y^2)(x^2 - y^2)$.
9. $(x^2 + 3)(x^2 - 3)$.
10. $(ab + cd)(ab - cd)$.
11. $(2x - y^2)(2x + y^2)$.
12. $(6 - 3b^2)(6 + 3b^2)$.
13. $(52)(48)$.
14. $(85)(75)$.
15. $(33)(27)$.

16. $(\frac{1}{2}x + \frac{1}{3}y)(\frac{1}{2}x - \frac{1}{3}y)$.
17. $(\frac{3}{4}x + \frac{2}{5}y)(\frac{3}{4}x - \frac{2}{5}y)$.
18. $(x^4 - y^4)(x^4 + y^4)$.
19. $(3a^2 - 11b^2)(3a^2 + 11b^2)$.

20. $(4a^2 - 7c^2)(4a^2 + 7c^2)$.
21. $(7x^2 + 3y^3)(7x^2 - 3y^3)$.
22. $(3a^3 - 4b^4)(3a^3 + 4b^4)$.
23. $\left(\frac{1}{3x} - \frac{7}{5y}\right)\left(\frac{1}{3x} + \frac{7}{5y}\right)$.
24. $\left(\frac{3x}{4} + \frac{2}{3y}\right)\left(\frac{3x}{4} - \frac{2}{3y}\right)$.
25. $\left(\frac{6x^2}{5y^2} - \frac{5y^2}{6x^2}\right)\left(\frac{6x^2}{5y^2} + \frac{5y^2}{6x^2}\right)$.
26. $\left(\frac{2a^3}{3b^2} + \frac{3x^2}{2y^3}\right)\left(\frac{2a^3}{3b^2} - \frac{3x^2}{2y^3}\right)$.

12. The Square of a Binomial. Let x represent the first term and y the second term of a binomial. We can verify by multiplication that

$$(x + y)^2 = x^2 + 2xy + y^2$$

and

$$(x - y)^2 = x^2 - 2xy + y^2.$$

These formulas for the square of $x + y$ and the square of $x - y$ are described in the statements which follow.

The square of a binomial is a trinomial. The terms of the trinomial are obtained by the following steps:

1. *Square the first term of the binomial.*
2. *Take twice the product of the terms of the binomial.*
3. *Square the second term of the binomial.*

The term determined by step 2 is given a plus sign if the terms of the binomial have like signs, and a minus sign if the terms have unlike signs.

Example 1. Write the product $(3x - 4y)^2$.
Solution.

$$\begin{aligned}(3x - 4y)^2 &= (3x)^2 + 2(3x)(-4y) + (-4y)^2 \\ &= 9x^2 - 24xy + 16y^2.\end{aligned}$$

Example 2. Write the product $(5a + b)^2$.
Solution.

$$(5a + b)^2 = 25a^2 + 10ab + b^2.$$

13. The Square of a Multinomial. By multiplication we may show that

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

and

$$(x + y + z + w)^2$$
$$= x^2 + y^2 + z^2 + w^2 + 2xy + 2xz + 2xw + 2yz + 2yw + 2zw.$$

These results indicate the correctness of the statement which follows.

The square of a multinomial is equal to the sum of the squares of the separate terms plus twice the product of each term by every term which follows it.

Example 1. Write the product $(a + 3b - 5)^2$.
Solution.

$$\begin{aligned}(a + 3b - 5)^2 &= a^2 + 9b^2 + 25 + 2(a)(3b) + 2(a)(-5) + 2(3b)(-5)\\ &= a^2 + 9b^2 + 25 + 6ab - 10a - 30b.\end{aligned}$$

Example 2. Write the product $(2x - y + 3z - 1)^2$.
Solution.

$$(2x - y + 3z - 1)^2$$

$$= 4x^2 + y^2 + 9z^2 + 1 - 4xy + 12xz - 4x - 6yz + 2y - 6z.$$

Exercise 6

Write the products by inspection:

1. $(x + 1)^2$.
2. $(x - 2)^2$.
3. $(x + 5)^2$.
4. $(2m - 1)^2$.
5. $(m + 3n)^2$.
6. $(m - 4n)^2$.
7. $(2x + 3y)^2$.
8. $(-x - 2y)^2$.
9. $(mn + 5)^2$.
10. $(7m - 5n)^2$.
11. $(xy - 3z)^2$.
12. $(5x - 6y)^2$.
13. 61^2.
14. 49^2.
15. 52^2.
16. $(2a^2 + b^2)^2$.
17. $(a^3 - b^3)^2$.
18. $(x^2 - 2y^2)^2$.
19. $(m^2n - 4)^2$.
20. $(-9 - 2n^2)^2$.
21. $(-4x^2 - 3y^2)^2$.
22. $(x - y + z)^2$.
23. $(x + 2y + z)^2$.
24. $(a - b - 2c)^2$.
25. $(a - 2b - 3)^2$.
26. $(4 - m - 3n)^2$.
27. $(x^2 + x + 1)^2$.
28. $(a^2 + b^2 + 2)^2$.
29. $(x + y + z - w)^2$.
30. $(x - y + z - w)^2$.
31. $(2a - b + 3c - d)^2$.
32. $(3m - n - p + 2q)^2$.
33. $(m - 2n + p - 3q)^2$.

Treat the factors in each problem as the sum and difference of two quantities and write the product:

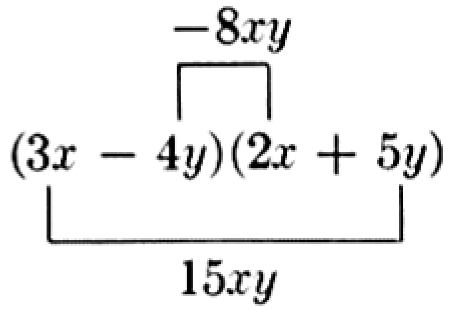
34. $(a + b + 1)(a + b - 1)$.
35. $(x - y + 2)(x - y - 2)$.
36. $(2x - y + 4)(2x - y - 4)$.
37. $(3m + 2n - 1)(3m + 2n + 1)$.
38. $(1 - 3m + n)(1 - 3m - n)$.
39. $(a + b + 1)(a - b - 1)$.
40. $(x + 5y - 1)(x - 5y + 1)$.
41. $(x^2 + 1 + x)(x^2 + 1 - x)$.

14. The Product of Two Binomials with Like Terms. The binomials $ax + by$ and $cx + dy$ have like terms. Their product is

$$(ax + by)(cx + dy) = acx^2 + (ad + bc)xy + bdy^2.$$

The following examples illustrate the application of this formula.

Example 1. Write by inspection the product of $3x - 4y$ and $2x + 5y$.
Solution. Write the binomials side by side.



$$-8xy + 15xy = 7xy$$

The product of the first terms of the binomials is $(3x)(2x) = 6x^2$. As indicated in the diagram, the product of the two inner terms is $(-4y)(2x) = -8xy$ and the product of the outer terms is $(3x)(5y) = 15xy$. The sum of these like terms is $7xy$. The product of the second terms of the binomials is $(-4y)(5y) = -20y^2$. Hence we have

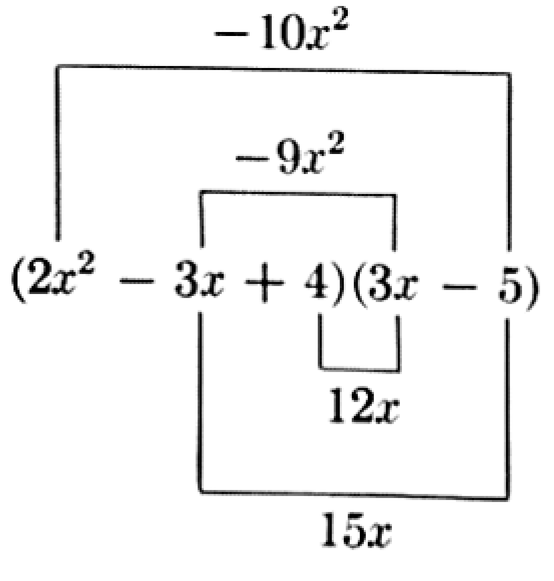
$$(3x - 4y)(2x + 5y) = 6x^2 + 7xy - 20y^2.$$

The middle term of this product is called the **cross product** term. Being the sum of two terms, it is the least simple to find. However, the idea is to determine mentally each term in a product of this kind and to write the result at once.

Example 2. Write the product $(4x - 3y)(x - 2y)$.
Solution. $(4x - 3y)(x - 2y) = 4x^2 - 11xy + 6x^2$.

Multiplication by inspection can be performed on multinomials involving more than two terms. This is illustrated in the next example.

Example 3. Multiply $2x^2 - 3x + 4$ by $3x - 5$.
Solution. We indicate the multiplication by placing the expressions side by side.



$$-9x^2 - 10x^2 = -19x^2$$

$$12x + 15x = 27x$$

The product of the first term of the trinomial and the first term of the binomial is $6x^3$. The product of the last term of the trinomial and the last term of the binomial is -20. The other kinds of terms in the product are an x^2 term and an x term. The diagram shows how to find these. Hence we have

$$(2x^2 - 3x + 4)(3x - 5) = 6x^3 - 19x^2 + 27x - 20.$$

Although we have explained, in words and by diagram, how the terms of the product are found, the suggested procedure is to write only the final result.

Exercise 7

Write the products by inspection:

1. $(x + 1)(x + 2)$.
2. $(x + 4)(x - 5)$.
3. $(x + 4)(x + 6)$.
4. $(x - 5)(x - 3)$.
5. $(x - y)(x - 2y)$.
6. $(2x - 1)(x - 2)$.
7. $(4 + 3m)(5 + 4m)$.
8. $(3m - 2n)(3m + n)$.
9. $(m - 3n)(m + 6n)$.
10. $(2a - 3b)(5a - b)$.
11. $(3a - 4b)(5a + b)$.
12. $(5a + 6b)(a - 5b)$.
13. $(7a + b)(-a - 6b)$.
14. $(-a - 5b)(5a - b)$.
15. $(5a - 7b)(3a - 8b)$.
16. $(x^2 - 4)(2x^2 - 1)$.
17. $(3x^2 - 8)(4x^2 + 3)$.
18. $(x^2 + 7)(x^2 + 6)$.

19. $(x^2 + x + 2)(x + 1)$.
20. $(x^2 - x - 3)(x + 2)$.
21. $(m^2 - 2m + 3)(m + 3)$.
22. $(2m^2 - 4)(3m + 5)$.
23. $(4m^2 - 5m)(2m - 3)$.
24. $(3m^2 + 3m - 1)(2m - 4)$.
25. $[(x + y) + 1][(x + y) + 3]$.
26. $[(x - y) - 3][(x - y) + 2]$.
27. $[(2x - y) + 4][(2x - y) - 6]$.
28. $[2(x + y) - 1][3(x + y) + 1]$.

15. The Cube of a Binomial. It may be verified by multiplication that

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad (1)$$

and

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3. \qquad (2)$$

A word description of these formulas follows.

The cube of a binomial has four terms determined by the following steps:

1. *Cube the first term of the binomial.*
2. *Take three times the square of the first term times the second term.*
3. *Take three times the first term times the square of the second term.*
4. *Cube the second term of the binomial.*

Example 1. Write the product $(2x - 5y)^3$.
Solution.

$$\begin{aligned}(2x - 5y)^3 &= (2x)^3 + 3(2x)^2(-5y) + 3(2x)(-5y)^2 + (-5y)^3\\ &= 8x^3 - 60x^2y + 150xy^2 - 125y^3.\end{aligned}$$

Example 2. Write the product $(3x + 2)^3$.
Solution.

$$(3x + 2)^3 = 27x^3 + 54x^2y + 36y + 8.$$

Exercise 8

Write the cube of each binomial:

1. $(x-1)^3$.
2. $(x-3)^3$.
3. $(x+4)^3$.
4. $(3+x)^3$.
5. $(2x+1)^3$.
6. $(x-2y)^3$.
7. $(5-x)^3$.
8. $(1-5x)^3$.
9. $(6x+1)^3$.
10. $(2a-3b)^3$.
11. $(-a-b)^3$.
12. $(-a-4)^3$.
13. $(5+ab)^3$.
14. $(a^2-3b)^3$.
15. $(2a^2+3b)^3$.
16. $(6mn-y)^3$.
17. $(3m+4xy)^3$.
18. $(x^3-y^3)^3$.

Supplementary Problems

Add the two expressions. Next subtract the second expression from the first:

1. $3a-4b;\ 4a+3b$.
2. $7x-2y;\ 5x-8y$.
3. $11x-3y;\ -4x+6y$.
4. $3c-4d;\ 7c+13d$.
5. $2x+3y-7;\ 4x-y+1$.
6. $4a-7b+6;\ a+b-4$.
7. $2m+n+5;\ m-6n+6$.
8. $x-3y-z;\ 3x+y+z$.
9. $2x+3y+5z-5;\ -3x-7y+z+10$.
10. $a-3b-7c+6;\ 4a-b+12c-9$.
11. $-7p-4q+7r-7;\ 3p+4q-6r-13$.
12. $6p+7q-5r-3;\ -9p-2q+2r+2$.

Add the three expressions. Next subtract the first expression from the sum of the second and third:

13. $7x+2y-5;\ 3x-6y+3;\ 4x-6y-3$.
14. $5a-b-4c;\ -7a+6b+3c;\ 3a+7b+4c$.
15. $-4x+5y-7z;\ 6x-7y+9z;\ -9x-3y+8z$.
16. $2a-2b-11c;\ -a+4b-6c;\ -2a+4b+8c$.
17. $4p+4q+5r;\ 3p-2q-7r;\ 2p-7q-7r$.
18. $x-7y-4z+5;\ 3x+3y+4z+6;\ x-2y-3z-4$.
19. $5a-5b-6c-7;\ -3a-2b+c+1;\ 7a+5b+3c-1$.
20. $p-2q-3r-4;\ 4p+5q-6r-7;\ -8p-4q+5r+3$.

Combine like terms:

21. $4x-3y-3x+7y$.
22. $3a+4b-5a-6b$.
23. $11a-4b+6b-7a$.
24. $4x-4y-3y+5x$.
25. $p-4q+7p-q$.
26. $2r+3s-4s-4r+s$.
27. $x-(y-2)+3$.
28. $(a-3b)+(-2a+5b)$.
29. $(2x-4y)+(-x-2y)$.
30. $(-a+4b)-(b-2a)$.
31. $(3m+n)-(m-3n)$.
32. $(8x+3y)-(-x+5y)$.
33. $2a-3b+7c+6-4-c+5b-6b+2a-4c$.
34. $6x-3z+4y+8-6-y+7x-10z+8$.
35. $2c+4y-7-3y-4c+2+5c-3+c$.

36. $xy + 2x - 3y + z - 3 + 2z - 6xy - 5x + 4y$.
37. $x - (2x - y - 3) + (3x + 4y - 5)$.
38. $(a - 2b + 3c) - (2a + 3b - 4c) + 2a - b$.
39. $(x - 3y - 4) - (3x + y + 3) + (4 - x - y)$.
40. $p - [(x - y) - (3x - 2y) + 4] - 3p + 6$.
41. $-[x - 6y - 2a - (6 - 4a) - 4] + 7x + 3y$.
42. $a - [(x - 7y + 3) - (7 - 9x - y) - 2] - [-4 - (x + y)]$.

Perform mentally the indicated operation:

43. $4m^2(3m - 2)$.
44. $2ab(3a - 2b)$.
45. $-ab^2(a^2 - 4b^2)$.
46. $(3x + y)(3x - y)$.
47. $(4a - 5)(4a + 5)$.
48. $(2m - 3n)(2m + 3n)$.

49. $(2x^2 - y^2)(2x^2 + y^2)$.
50. $(3mn + 4)(3mn - 4)$.

51. $(4x^2y - 3)(4x^2y + 3)$.
52. $(x + 1)(x + 3)$.
53. $(a - 1)(a - 2)$.
54. $(a + 4)(a - 5)$.
55. $(2m - n)(m + 2n)$.
56. $(4m + 3n)(m - n)$.

57. $(2x + 3y)(3x - 2y)$.
58. $(2 - 5x)(1 + 4x)$.
59. $(2m - 3n)(2m - 3n)$.
60. $(5x - 4y)(6x + 5y)$.
61. $(a^2 + a + 1)(a + 1)$.
62. $(a^2 - 2a + 1)(2a + 1)$.
63. $(2x + 3)(x^2 + 3x - 4)$.
64. $(x - 1)(2x^2 - 4x + 1)$.
65. $(3m^2 + m - 4)(2m - 3)$.
66. $(6m^2 + m + 3)(2m - 3)$.

Perform the indicated operations:

67. $2x(x - 2y) - 3x(y - 4x)$.
68. $3y(4x - 2y) + 4x(3x + 2y)$.
69. $4m(m - 2n - 3) - 3n(2m - n)$.
70. $ab(a - b + 1) - a(2ab - b^2 - b)$.
71. $(3x^2 - 2)(4x^2 - x - 5)$.
72. $(5x^2 + 3)(x^2 - 6x - 7)$.
73. $(a + 2b + 1)(a + 2b - 1)$.
74. $(2a - 3b - 2)(2a + 3b + 2)$.
75. $(m^2 + 2m - 5)(2m^2 - 3m + 4)$.
76. $(6m^2 + 3m + 2)(4m^2 - m + 3)$.
77. $(2x^2 - 3x + 7)(3x^2 + 4x - 5)$.
78. $(3x - 4)(2x^3 - 4x^2 - 6x - 2)$.
79. $(y^3 - 7y^2 - 3y + 4)(2y + 3)$.
80. $(3x - 4y + 2)(x^2 - 2xy - y^2)$.

81. $(a^3 - 2a^2 + 3a + 5)(2a^2 + a^2 - 4a + 6)$.
82. $(m^2 - mn + n^2 + 1)(2m^2 + 3mn - 2n^2 - 4)$.

Divide the first expression by the second:

83. $3x^2 + 7xy + 2y^2;\ x + 2y$.
84. $8a^2 - 6ab - 9b^2;\ 2a - 3b$.
85. $10x^2 - 7xy - 12y^2;\ 5x + 4y$.
86. $6b^2 + 13bc - 15c^2;\ 6b - 5c$.
87. $6x^3 - x^2 - 3x + 18;\ 2x + 3$.
88. $3 - 17x + 9x^2 + 5x^3;\ 1 - 5x$.

89. $2x^3 - x^2y + xy^2 + y^3;\ 2x + y$.
90. $2x^4 - 5x^3 + 2x^2 + 13x - 12;\ x^2 - 3x + 4$.
91. $6m^4 + 5m^3 - 10m^2 + 7m - 2;\ 3m^2 - 2m + 1$.
92. $x^5 - 2x^4 - 3x^2 + 2x - 4;\ x^3 - x^2 - 2x - 4$.
93. $6x^5 - 2x^4 - 11x^3 - 8x^2 + 7x - 12;\ 3x^2 - x - 4$.
94. $5x^5 - 4x^4 + 4x^3 + 16x^2 - 12x - 3;\ 5x + 1$.

Raise to the power indicated:

95. $(x + 2y)^2$. **96.** $(3x - 4)^2$. **97.** $(2x - 5y)^2$.
98. $(4 + 2y)^2$. **99.** $(7 + 3y)^2$. **100.** $(x^2 - 2y)^2$.
101. $(3m - n^2)^2$. **102.** $(mn + 7)^2$. **103.** $(2x^2 - y^3)^2$.
104. $(3 - x)^3$. **105.** $(2a + b)^3$. **106.** $(1 - 2x)^3$.
107. $(x^2 - 1)^3$. **108.** $(2x^3 + y)^3$. **109.** $(mn + 2a)^3$.
110. $(x - y - z)^2$. **111.** $(2a + b - c)^2$. **112.** $(2m - 3n + 1)^2$.

113. $(a - b + c - d)^2$. **114.** $(2a + 3b - c - 3d)^2$.

Chapter Two

FACTORING

16. Introduction. A considerable part of Chapter One dealt with finding the products of factors. In this chapter we shall consider the problem of finding the factors when the product is given. This process, called **factoring,** is used extensively in performing algebraic operations. Factoring is usually restricted to expressions of a special kind, which we now discuss.

17. Rational Integral Expressions. A rational integral expression in x is the sum of terms like kx^n, where n is a positive integer and k is a coefficient free of x. In addition to the terms involving x, there may also be terms composed of other numbers called constant terms. The word **polynomial** is used with the same meaning as rational integral expression. The following are rational integral expressions, or polynomials:

$$3x + 4; \quad \tfrac{3}{4}x^3 - \tfrac{1}{2}x^2 - 5x + 7; \quad 3x^4 - 5x^2 + 7x.$$

We may have polynomials in two, three, or more letters. The expressions

$$5x^3 - 2x^2y + 3xy^2 - 4y + 5 \quad \text{and} \quad \tfrac{1}{4}x^4y + 5xy^2 - 2y^3 - y^4$$

are polynomials in x and y.

The **degree** of each term of a polynomial in x is the exponent of x in that term. The **degree of the polynomial** is defined as the greatest exponent of x in the expression. For example, the terms of

$$7x^4 + 6x^3 - x^2 - 5x + 3$$

are, respectively, of degrees four, three, two, and one. The constant term 3 may be regarded as of zero degree. The expression is of degree four.

In a polynomial of more than one letter the degree of any term is the sum of the exponents of the letters in that term. And the term or terms of highest degree determine the degree of the expression. The terms of

$$2x^3y^2z^2 - 3x^2yz + xy - x + 5$$

are, in order, of degrees seven, four, two, one, and zero. The polynomial is of seventh degree.

Factoring of polynomials is usually restricted to those in which each coefficient is a rational number. That is, each coefficient is equal to the quotient of two integers. We say that a polynomial is factored when it is expressed as the product of polynomials each of which has rational coefficients. And the polynomial is completely factored if none of its factors can be factored. Each factor, then, being expressible as a product only as 1 times itself or minus 1 times its negative, is said to be **prime.**

18. Common Factors. If each term of a polynomial has a common factor, the polynomial may be written at once as the product of two factors, one of which is the common factor.

Example 1. Factor $ax + ay - az$.

Solution. Each term has the factor a. Then if we divide the given expression by a, we obtain the other factor $x + y - z$. Hence,

$$ax + ay - az = a(x + y - z).$$

Example 2. Factor $6ax + 3bx + 9cx$.

Solution. Each term has $3x$ as a factor, and

$$6ax + 3bx + 9cx = 3x(2a + b + 3c).$$

The common factor of each term of an expression may have more than one term. Now let us consider the case in which the common factor is a binomial.

Example 3. Factor $x(a + b) + y(a + b)$.

Solution. Here one term is $x(a + b)$ and the other $y(a + b)$; each of the terms has $(a + b)$ as a factor. Then dividing the given expression by $a + b$, we obtain $x + y$, which is the other factor. Hence,

$$x(a + b) + y(a + b) = (a + b)(x + y).$$

The factoring could also be done by first writing one letter to stand for $a + b$. Thus,

$$x(a + b) + y(a + b) = xc + yc = c(x + y) = (a + b)(x + y).$$

19. The Difference of Two Squares. We have noticed that the product of the sum and difference of two numbers is equal to the difference of their squares. That is, $(x + y)(x - y) = x^2 - y^2$. Reversing the order of writing these equal quantities, we have the factoring formula

$$\mathbf{x^2 - y^2 = (x + y)(x - y).}$$

From this formula we may state the rule for factoring the difference of the squares of two quantities.

The difference of the squares of two numbers is equal to the product of the sum and difference of the numbers.

Example 1. Factor $4a^2 - 9b^2$.
Solution.

$$4a^2 - 9b^2 = (2a)^2 - (3b)^2 = (2a + 3b)(2a - 3b).$$

The intermediate step is written to make it perfectly clear that the expression is really the square of one quantity minus the square of another. It is usually desirable to omit this step and write the factors at once.

Example 2. Factor $4ax^2 - 16ay^2$.
Solution. Remove the common factor $4a$ first. Then we have

$$4ax^2 - 16ay^2 = 4a(x^2 - 4y^2) = 4a(x + 2y)(x - 2y).$$

The quantities which are squared may be monomials, as in the examples above, or they may be of more than one term.

Example 3. Factor $(a + b)^2 - (c - 2d)^2$.
Solution. We have the square of $a + b$ minus the square of $c - 2d$. Hence, one factor is $(a + b) + (c - 2d) = a + b + c - 2d$, and the other is $(a + b) - (c - 2d) = a + b - c + 2d$. Then we have

$$(a + b)^2 - (c - 2d)^2 = (a + b + c - 2d)(a + b - c + 2d).$$

Exercise 9

Factor each expression completely:

1. $ax + ay$.
2. $3x + 9y$.
3. $6x - 12xy$.
4. $5mn - 10m^2$.
5. $8m^2 - 16m$.
6. $20m^2 - 5m$.
7. $4x^2 - xy + x$.
8. $3x^2 - 3xy + 6xy^2$.
9. $x^2y - xy^2 + xy$.

10. $3(x + 4) + a(x + 4)$.
11. $p(m - n) + q(m - n)$.
12. $m(a + b) - (a + b)$.
13. $a(x^2 - x - 1) - b(x^2 - x - 1)$.
14. $(3a - 2b)(x + 6y) + (x + 6y)$.
15. $(x - 2y) + (x + y)(x - 2y)$.

16. $(m - 3n)(2m + 3n) + (3m - n)(2m + 3n)$.
17. $(4x - y)(x - 2y) + 3(4x - y) + (x + a)(4x - y)$.

18. $9x^2 - y^2$.
19. $x^2 - 16$.
20. $1 - 25x^2$.
21. $4a^2 - 25b^2$.
22. $8a^2 - 32b^2$.
23. $2ax^2 - 18ay^2$.
24. $36 - m^2n^2$.
25. $m^3 - 16m$.
26. $4m^2n^2 - 9c^2$.
27. $3a - 27ab^2$.
28. $(a - 1)^2 - b^2$.
29. $a^2 - (b + 2)^2$.
30. $16x^2 - (2y - 3)^2$.
31. $(x - y)^2 - 9a^2$.
32. $4x^2y^2 - (z + 1)^2$.

33. $(m + n)^2 - (a + b)^2$.
34. $(m + n)^2 - (a - b)^2$.
35. $(2m - 1)^2 - (n - 1)^2$.
36. $x^2(2a - 3) - 4y^2(2a - 3)$.

37. $4x^2(p + q) - 64(p + q)$.
38. $(3x + 4)(m^2 - n^2) - (2x - 3)(m^2 - n^2)$.
39. $(2x^2 - 8)(y^2 - a^2) + (2x^2 - 8)(a^2 - x^2)$.

Factor and then multiply the two factors:

40. $(48)^2 - (22)^2 = (48 + 22)(48 - 22) = 70(26) = 1820$.

41. $(52)^2 - (48)^2$.
42. $(56)^2 - (46)^2$.
43. $(55)^2 - (35)^2$.

20. The Sum or Difference of Two Cubes. If we divide $x^3 + y^3$ by $x + y$, we obtain the quotient $x^2 - xy + y^2$. Hence,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

A word statement of this formula follows.

One factor of the sum of the cubes of two quantities is the sum of the quantities, and the other factor is the square of the first quantity minus the product of the two quantities plus the square of the second.

Similarly, it may be verified that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

One factor of the difference of the cubes of two quantities is the difference of the quantities, and the other factor is the square of the first quantity plus the product of the two quantities plus the square of the second.

Example 1. Factor $a^3b^3 + c^3$.
Solution.

$$a^3b^3 + c^3 = (ab)^3 + c^3 = (ab + c)(a^2b^2 - abc + c^2).$$

Example 2. Factor $27x^3 - y^3$.
Solution.

$$27x^3 - y^3 = (3x - y)(9x^2 + 3xy + y^2).$$

Example 3. Factor $(x + y)^3 + z^3$.
Solution.

$$(x + y)^3 + z^3 = [(x + y) + z][(x + y)^2 - (x + y)z + z^2]$$
$$= (x + y + z)(x^2 + 2xy + y^2 - xz - yz + z^2).$$

Exercise 10

Factor:

1. $x^3 + 27$.
2. $8 + x^3$.
3. $8x^3 + 1$.
4. $x^3 - 64$.
5. $1 - 8x^3$.
6. $125 - x^3$.
7. $16a^3 + 2b^3$.
8. $8a^3 - 27b^3$.
9. $27a^3b^3 + c^3$.
10. $ab^3 + 27a^4$.
11. $3a^3b^3 - 24$.
12. $64a^3 - 27b^3$.
13. $(x + y)^3 - z^3$.
14. $(x - 2y)^3 + 1$.
15. $x^3 + (2y - z)^3$.
16. $8(x + 1)^3 + y^3$.
17. $27 - (x - y)^3$.
18. $1 - 8(x + y)^3$.
19. $m^3 - (2n - 3)^3$.
20. $(m + n)^3 - (a - b)^3$.
21. $(m - n)^3 + (a + b)^3$.

21. Trinomials. We have seen how to find by inspection the square of a binomial and the product of two binomials with like terms. In each of these cases the product is a trinomial. Our problem now is to factor trinomials of this nature.

From Section 12, we have

$$x^2 + 2xy + y^2 = (x + y)^2$$

and

$$x^2 - 2xy + y^2 = (x - y)^2.$$

An examination of these formulas will furnish a way for determining if a trinomial is the square of a binomial. The middle term of the first trinomial is twice the product of two quantities and the other terms are the squares of the quantities. This, therefore, is a test for determining if a trinomial is the square of the sum of two quantities. The second trinomial is minus twice the product of two quantities plus the sum of the squares of the quantities, and this means that the trinomial is the square of the difference of two quantities.

Example 1. Factor $25x^2 + 30x + 9$.

Solution. The first term is the square of $5x$ and the last term is the square of 3. The middle term is twice the product of $5x$ and 3. Hence the trinomial is a perfect square, and we write

$$25x^2 + 30x + 9 = (5x + 3)^2.$$

Example 2. Factor $9x^2 - 12xy + 4y^2$.

Solution. The first term is $(3x)^2$ and the last $(2y)^2$. The middle term is minus twice the product of $3x$ and $2y$. This means that the trinomial is the square of the difference of $3x$ and $2y$. Hence we have

$$9x^2 - 12xy + 4y^2 = (3x - 2y)^2.$$

We next illustrate the case of factorable trinomials which are not perfect squares. Factoring in this case is less simple than that of perfect square trinomials. However, the following examples show the procedure.

Example 3. Factor $x^2 - 5x + 6$.

Solution. If x is used as the first term of each factor, the product will contain x^2. The product of the other terms must be 6. This value is the product of 1 and 6, of -1 and -6, of 2 and 3, and of -2 and -3. Observing the middle term of the given trinomial, we see that the terms whose product is 6 must have the sum -5. Only the pair -2 and -3 have the required sum. Hence we have

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

Example 4. Factor $8x^2 - 2xy - 15y^2$.

Solution. For the first terms of the factors x and $8x$ or $2x$ and $4x$ appear as possibilities. The second terms must have $-15y^2$ as their product. The choices for this product are $-y$ and $15y$, y and $-15y$, $-3y$ and $5y$, and $3y$ and $-5y$. But the sum of the cross products (the first term of each factor times the second term of the other) must yield $-2xy$. By trying the various combinations it may be determined that $2x - 3y$ and $4x + 5y$ are the factors. Hence,

$$8x^2 - 2xy - 15y^2 = (2x - 3y)(4x + 5y).$$

Exercise 11

Factor:

1. $x^2 + 4x + 4$.
2. $x^2 - 6x + 9$.
3. $4x^2 + 12x + 9$.
4. $16x^2 - 24x + 9$.
5. $1 - 8xy + 16x^2y^2$.
6. $25 - 30x + 9x^2$.
7. $9a^2 + 12ab + 4b^2$.
8. $16a^2 + 8ab + b^2$.
9. $a^2 - 2ab + b^2$.
10. $a^2 + 6a + 8$.
11. $a^2 + 7a + 6$.
12. $a^2 + 5a + 4$.
13. $m^2 - 7m + 6$.
14. $m^2 - 9m + 18$.
15. $m^2 - 9m + 20$.
16. $m^2 - m - 12$.
17. $m^2 + 2m - 15$.
18. $m^2 + 5m - 14$.
19. $a^2 - 4ab - 21b^2$.
20. $a^2 - 5ab - 36b^2$.
21. $a^2 + 11ab + 18b^2$.
22. $2x^2 + x - 3$.
23. $3x^2 + 4x - 4$.
24. $4x^2 + 23x - 6$.
25. $10x^2 - 13x - 3$.
26. $4x^2 - 4x - 15$.
27. $24 - 10x - 21x^2$.
28. $21 - 25x - 4x^2$.
29. $3x^2y^2 - 11xy + 8$.
30. $6x^2 - 11xy - 7y^2$.
31. $12x^3 - 11x^2 + 2x$.
32. $x^4 + 13x^3 + 42x^2$.
33. $4x^2y^2 - 5xy^3 - 6y^4$.
34. $a^4 + 6a^2 + 9$.
35. $9 - 6a^2 + a^4$.
36. $4a^4 + 8a^2 + 4$.
37. $a^4 - 3a^2 - 4$.
38. $3a^4 - 10a^2 - 8$.
39. $2a^4 - a^2 - 1$.
40. $a^4 - 8a^2 + 16$.
41. $81 + 18a^2 + a^4$.
42. $25a^4 + 20a^2 + 4$.
43. $6x^4 - 5x^2 - 6$.
44. $4x^4 - 17x^2 + 4$.
45. $10 - 11x^2 - 6x^4$.
46. $(m - n)^2 + 2(m - n) + 1$.
47. $(2m - n)^2 - 4(2m - n) + 4$.
48. $(m - 4n)^2 + 3(m - 4n) - 10$.
49. $(m + n)^2 - (m + n) - 2$.
50. $3(x - y)^2 - 7(x - y) + 2$.
51. $4(x + 2y)^2 + 4(x + 2y) - 3$.

Factor the expressions **52–55.** Each is the square of a trinomial.

52. $x^2 + y^2 + 1 + 2xy + 2x + 2y$.
53. $x^2 + y^2 + 4 - 2xy + 4x - 4y$.
54. $x^2 + 4y^2 + 4 + 4xy - 4x - 8y$.
55. $x^2 + y^2 + 9z^2 - 2xy - 6xz + 6yz$.

Factor the expressions **56–59.** Each is the cube of a binomial.

56. $x^3 + 6x^2 + 12x + 8$.
57. $8y^3 - 36y^2 + 54y - 27$.
58. $a^3 - 18a^2 + 108a - 216$.
59. $125a^3 + 150a^2 + 60a + 8$.

22. A Common Factor by Grouping. Sometimes a multinomial whose terms contain no common factor can be separated into groups of terms which have a common factor. The common factor of the groups is a factor of the multinomial.

Example 1. Factor $3x + 3y + ax + ay$.

Solution. The terms contain no common factor. But a common factor results by the grouping of the first two and the last two terms. Thus we get

$$\begin{aligned} 3x + 3y + ax + ay &= (3x + 3y) + (ax + ay) \\ &= 3(x + y) + a(x + y) \\ &= (x + y)(3 + a). \end{aligned}$$

The factors could also be obtained by grouping the first and third terms and the second and fourth terms.

Example 2. Factor $x^2 - y^2 - 2x + 2y$

Solution.

$$\begin{aligned} x^2 - y^2 - 2x + 2y &= (x^2 - y^2) + (-2x + 2y) \\ &= (x + y)(x - y) - 2(x - y) \\ &= (x - y)(x + y - 2). \end{aligned}$$

23. Multinomials Reducible to the Difference of Two Squares. Some multinomials not in the form of the difference of two squares may be so expressed by a proper grouping of the terms. We shall illustrate two slightly different devices which are used for this purpose. In one case the terms as given can be separated into the difference of two squares; in the other the multinomial is first altered by the addition and subtraction of a term.

Example 1. Factor $4 - x^2 + 2xy - y^2$.

Solution. The last three terms with their signs reversed may be recognized as the square of $x - y$. We make the change in signs by the use of parentheses preceded by a minus sign. Thus we have

$$\begin{aligned} 4 - x^2 + 2xy - y^2 &= 4 - (x^2 - 2xy + y^2) \\ &= 4 - (x - y)^2 \\ &= (2 + x - y)(2 - x + y). \end{aligned}$$

Example 2. Factor $x^4 - 6x^2 + 1$.

Solution. The terms as given cannot be grouped into the difference of two squares. The desired grouping can be achieved, however, by the introduction of a new term. The given trinomial would be a perfect square if the middle term were $2x^2$ or $-2x^2$. Hence the addition of either $8x^2$ or $4x^2$ yields a trinomial which is a perfect square. If a term is added, it must also be subtracted in order that the new expression shall be equal to the original one. It becomes evident, therefore, that the term chosen for addition and subtraction should itself be a perfect square. On this basis $8x^2$ must be rejected. Adding $4x^2$ and then subtracting $4x^2$, we get

$$\begin{aligned} x^4 - 6x^2 + 1 &= x^4 - 2x^2 + 1 - 4x^2 \\ &= (x^2 - 1)^2 - (2x)^2 \\ &= (x^2 - 1 + 2x)(x^2 - 1 - 2x). \end{aligned}$$

It should be understood that the device of adding and subtracting a term, for the purpose of factoring, works only if (a) the expression to be factored becomes a perfect square by the addition, not the subtraction, of a term, (b) the term added, and also subtracted, is a perfect square.

Exercise 12

Factor:

1. $ax + ay + 4x + 4y$.
2. $4x + 12 + xy + 3y$.
3. $5x - 5y + ax - ay$.
4. $12xy + 9cy + 3cx + 4x^2$.
5. $ax - ay + 2cx - 2cy$.
6. $9x + 3y - 3xc - cy$.
7. $9xy - 3ax + 3y - a$.
8. $3xy + 9ax - ay - 3a^2$.
9. $m^3 - 2m^2 - 3m + 6$.
10. $4 - 2m - 4m^2 + 2m^3$.
11. $am^3 - 3m^2 - am + 3$.
12. $2am^3 - 8am + 2m^2 - 8$.
13. $ax - ay - az + bx - by - bz$.
14. $2ax + 3ay - az + 4x + 6y - 2z$.
15. $m^2 + 2m + 1 + am + a$.
16. $m^2 - 4mn + 4n^2 - 3m + 6n$.
17. $9m^2 - 6m + 1 - 3am + a$.
18. $m^2 + 2mn + n^2 + 4m + 4n$.
19. $a^2 - b^2 + 5a + 5b$.
20. $a^2 - 4b^2 - 3a + 6b$.
21. $a^2 + 2a - 3 + ab + 3b$.
22. $2a^2 - 3a - 2 + 4ab + 2b$.
23. $x^3 + y^3 + x + y$.
24. $x^3 - y^3 - x + y$.
25. $a^3 - b^3 - a^2 + b^2$.
26. $a^3 + b^3 - a^2 + ab - b^2$.
27. $a^3 + b^3 - a^2 - 2ab - b^2$.
28. $a^2 - 2ab + b^2 - 1$.

29. $m^2 + 6mn + 9n^2 - 9.$
30. $m^2 - 8m + 16 - y^2.$
31. $m^2 - 9 + 12n - 4n^2.$
32. $m^2 + 4n^2 - 16c^2 - 4mn.$
33. $16 - m^2 - 2mn - n^2.$
34. $x^2 - 2x + 1 - y^2 - 2yz - z^2.$
35. $9y^2 - 4x^2 - 12x - 9.$
36. $x^2 + 4x + 4 - y^2 + 2yz - z^2.$
37. $9x^2 - 6x + 1 - y^2 - 4yz - 4z^2.$
38. $x^2 - 2xy + y^2 - a^2 + 2ab - b^2.$
39. $x^4 + 4.$
40. $64a^4 + 1.$
41. $x^4 - 3x^2 + 1.$
42. $x^4 - 6x^2 + 25.$
43. $4a^4 + 8a^2b^2 + 9b^4.$
44. $x^4y^4 + x^2y^2 + 1.$
45. $y^4 - 17y^2z^2 + 16z^4.$
46. $25y^4 - 21y^2 + 4.$
47. $49y^4 - 15y^2 + 1.$
48. $36y^4 + 11y^2 + 1.$

24. Binomials of the Form $x^n - y^n$ and $x^n + y^n$. The binomial $x^n - y^n$, where n is a positive even integer, is factorable as the difference of two squares. For the case in which n is divisible by 3, the binomial may be factored as the difference of two cubes. If n is divisible by both 2 and 3, the factoring is facilitated by treating $x^n - y^n$ as the difference of two squares rather than as the difference of two cubes.

The binomial $x^n + y^n$, where n is a positive multiple of 3, is factorable as the sum of two cubes.

Illustrations:

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y).$$

$$\begin{aligned} x^6 - y^6 &= (x^3 + y^3)(x^3 - y^3) \\ &= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2). \end{aligned}$$

$$x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4).$$

$$\begin{aligned} x^9 + y^9 &= (x^3 + y^3)(x^6 - x^3y^3 + y^6) \\ &= (x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6). \end{aligned}$$

It will be seen in Chapter Fifteen that $x^n - y^n$ and $x^n + y^n$ may be factored for certain other positive integral values of n than those here considered.

Binomials which have terms of different degrees are sometimes expressible as the difference of two squares or cubes, or as the sum of two cubes.

Illustrations:

$$x^6 - y^4 = (x^3 + y^2)(x^3 - y^2).$$

$$x^9 + y^6 = (x^3 + y^2)(x^6 - x^3y^2 + y^4).$$

$$\begin{aligned} x^6 - y^{12} &= (x^3 + y^6)(x^3 - y^6) \\ &= (x + y^2)(x^2 - xy^2 + y^4)(x - y^2)(x^2 + xy^2 + y^4). \end{aligned}$$

Exercise 13

Factor:

1. $x^4 - 1$.
2. $16 - x^4$.
3. $1 - 81x^4$.
4. $625x^4 - 1$.
5. $81x^4 - 16y^4$.
6. $256 - x^8$.
7. $m^6 - 1$.
8. $64 - m^6$.
9. $m^6 - 8$.
10. $64m^6 + 1$.
11. $m^6 + 64n^6$.
12. $m^6n^6 + 1$.
13. $x^8 - 16$.
14. $16x^8 - 9^4$.
15. $x^{10} - y^4$.
16. $x^4y^6 - 16$.
17. $x^9 + y^9$.
18. $x^{12} - y^{12}$.
19. $x^9 + y^6$.
20. $x^6 + y^{12}$.
21. $x^9 - 27y^3$.

25. The Lowest Common Multiple. A polynomial which is exactly divisible by two or more polynomials is called a **common multiple** of the latter polynomials. The common multiple determined in the following way is called the **lowest common multiple (L.C.M.)**:

1. Factor each of the given polynomials.

2. Form a product by using all the different prime factors of the polynomials, giving each prime factor the largest exponent which that factor has in any of the polynomials.

Example 1. Find the L.C.M. of $24xy^2$, $18x^2y^3$, and $36xyz$.

Solution. We factor each of the numerical coefficients.

$$24xy^2 = 2^3 \cdot 3 \cdot x \cdot y^2$$

$$18x^2y^3 = 2 \cdot 3^2 \cdot x^2 \cdot y^3$$

$$36xyz = 2^2 \cdot 3^2 \cdot x \cdot y \cdot z$$

The prime factors in the given expressions are 2, 3, x, y, and z. We give each of these factors the greatest exponent appearing in any of the factored expressions and form the product

$$2^3 \cdot 3^2 \cdot x^2 \cdot y^3 \cdot z = 72x^2y^3z.$$

This result is the L.C.M. of the given expressions.

Example 2. Find the L.C.M. of $6x^2 + 3x - 3$, $7x^2 - 14x + 7$, and $12x^2 - 12$.

Solution. We first factor each polynomial completely.

$$6x^2 + 3x - 3 = 3(2x^2 + x - 1) = 3(2x - 1)(x + 1)$$

$$7x^2 - 14x + 7 = 7(x^2 - 2x + 1) = 7(x - 1)^2$$

$$12x^2 - 12 = 12(x^2 - 1) = 2^2 \cdot 3(x + 1)(x - 1).$$

Selecting the proper power of each prime factor, we get

$$2^2 \cdot 3 \cdot 7(2x - 1)(x + 1)(x - 1)^2 = 84(2x - 1)(x + 1)(x - 1)^2,$$

which is the lowest common multiple.

The L.C.M. may be expressed in factored form or the actual multiplication may be carried out. The factored form, however, is usually more convenient.

Exercise 14

Find the L.C.M., leaving each result in the factored form except for the numerical coefficients:

1. $6xy^2, 4x^3y, 36xy.$
2. $18xy, 20x^2y^2, 30x^3y.$
3. $10xyz, 21x^2yz, 70xyz^3.$
4. $60xy, 36x^3, 84y^2z^4.$
5. $4x - 4, x^2 - 1.$
6. $4x - 2, 4x^2 - 1.$
7. $2x^2 + 2x - 4, 3x^2 + 7x + 2.$
8. $6x^2 + 9x - 6, 8x^2 - 2.$
9. $m^2 - m - 6, 2m^2 - 7m + 3.$
10. $2m^2 - m - 3, 4m^2 - 12m + 9.$
11. $m^2 - n^2, m^2 - 2mn + n^2.$
12. $2m^2 + 3m - 2, 4m^2 - 1.$
13. $m^2 + m, m^3 - m, m^2 - 1.$
14. $2m^2n^2 - 8, 6m^2n^2 - 9mn - 6.$
15. $2x^2 - 7x + 6, 3x^2 - 8x + 4.$
16. $4x^2 - 15x - 4, 8x^2 - 2x - 1.$
17. $x^3 + 1, x^3 - 1, x^4 + x^2 + 1.$
18. $x^4 + 4x^2 + 16, x^3 + 8.$
19. $x^2 - 16, x^3 - 64, 4x^2 - 16x.$
20. $x^4 - 8x^2 - 9, 4x^3 - 36x.$

21. $2m^2 - 5mn + 2n^2, n^2 - 4mn + 4m^2, 8m^3 - n^3.$
22. $8m^3 - 27n^3, 16m^3 + 24m^2n + 36mn^2, 4m^2 - 12mn + 9n^2.$
23. $m^4 - n^4, m^3 - m^2n + mn^2 - n^3, m^3 + m^2n - mn^2 - n^3.$
24. $x^2 - y^2 - 2yz - z^2, x^2 + 2xy + y^2 - z^2, x^2 - 2xz + z^2 - y^2.$
25. $x^2 - 2xz + z^2 - 4y^2, x^2 + 4xy + 4y^2 - z^2, x^2 - 4y^2 - 4yz - z^2.$
26. $12x^2 - 25xy + 12y^2, 9x^2 - 24xy + 16y^2, 27x^3 - 64y^3.$
27. $27x^3 + 8y^3, 6x^2 + xy - 2y^2, 9x^2 + 12xy + 4y^2.$
28. $9x^2 + 9xy - 108y^2, 24x^2 - 6xy - 3y^2, x^2 - 5xy + 6y^2.$

Supplementary Problems

Factor the following expressions:

1. $x^2 - 3xy + x.$
2. $2ab^2 - 4ab + 2a^2b.$
3. $6xy - 3x^2y + 9xy^2.$
4. $9a^2 - 1.$
5. $4x^2 - 9y^2.$
6. $36m^2 - 25n^2.$
7. $8m^2 - 2n^2.$
8. $3xy^2 - 12x^3.$
9. $(x + y)^2 - 9.$
10. $8 + a^3.$
11. $y^3 - 64.$
12. $1 - 27x^3.$
13. $8m^3 + 27n^3.$
14. $3ap^3 + 24aq^3.$
15. $ax^4 - 8axy^3.$
16. $1 - (x + y)^3.$
17. $(x + y)^3 + 1.$
18. $16x^3 + 54y^3z^3.$
19. $x^2 - 9x + 20.$
20. $x^2 - 2x - 24.$
21. $x^2 - 11x + 30.$
22. $4 - 4a + a^2.$
23. $m^2n^2 + 13mn + 42.$
24. $2x^2 - 5x + 2.$
25. $2x^2 + 7x + 6.$
26. $6a^2 + a - 12.$
27. $15a^2 - a - 2.$

28. $x^4 - 3x^2 - 4$.
29. $x^6 + x^3 - 2$.
30. $12m^2 + 10mn - 12n^2$.
31. $9x^2 + 6xy + y^2$.
32. $4a^2 - 20ab + 25b^2$.
33. $4a^2 - 12ab + 9b^2$.
34. $4(a - b) - a(a - b)$.
35. $x(y + 2) - y(y + 2)$.
36. $x^2(m + n) - 4y^2(m + n)$.
37. $a^3(x - y) - 8b^3(x - y)$.
38. $8a + 2ab - 4ay - aby$.
39. $6ab - 15a + 4bc - 10c$.
40. $x^2 - y^2 + 3x - 3y$.
41. $x^2 + 4xy + 4y^2 + x + 2y$.
42. $2x^3 + 6x^2 - 3x - 9$.
43. $x^2 - x - 2 + xy - 2y$.
44. $m^3 - n^3 - m^2 + 2mn - n^2$.
45. $4x^2 - 4xy + y^2 - 9$.
46. $25 - a^2 + 6a - 9$.
47. $9x^2 - 6xy + y^2 - 16a^2$.
48. $6x^2 - 5xy + 18xz - 15yz$.
49. $2x^2 - xy + xz - 2x + y - z$.
50. $x^3 - 3x^2 + 3x - 1$.
51. $x^3 + 6x^2 + 12x + 8$.
52. $8a^3 + 36a^2 + 54a + 27$.
53. $1 - 12x + 48x^2 - 64x^3$.
54. $x^2 + y^2 + 4 - 2xy + 4x - 4y$.
55. $m^2 + n^2 + 9 + 2mn - 6m - 6n$.
56. $x^2 - 4xy + 4y^2 - a^2 - 2ab - b^2$.
57. $4x^2 + 4xy + y^2 - 9x^2 + 12x - 4$.

58. $16x^4 - y^4$.
59. $25a^4 - 16$.
60. $m^6 - n^6$.
61. $8m^6 + n^6$.
62. $m^6 - y^3$.
63. $4m^6 - y^4$.
64. $x^4 + x^2 + 1$.
65. $m^4 + 3m^2 + 4$.
66. $4a^4 + 3a^2 + 1$.
67. $4x^4 - 21x^2y^2 + 9y^4$.
68. $9x^4 - 31x^2y^2 + y^4$.
69. $x^4 - 14x^2y^2 + 25y^4$.

70. $(x + y)^2 + 3(x + y) - 4$.
71. $(2x + y)^2 - (2x + y) - 6$.
72. $2(a - b)^2 + 3(a - b) - 2$.
73. $9(m - n)^2 - 9(m - n) + 2$.

Chapter Three

FRACTIONS

26. Algebraic Fractions. Fractions involving literal numbers arise in algebra. The indicated divisions in the problems of Exercise 4 are illustrations of fractions of this kind. A fraction with a literal number is sometimes called an **algebraic fraction** to distinguish it from the fractions of arithmetic. Since division by zero is forbidden, we shall assume that the letters in a fraction may not take values which make a denominator equal to zero.

The operations of addition, subtraction, multiplication, and division are applied to algebraic fractions just as in arithmetic. In connection with these operations we shall make use of a fundamental principle which applies both to common fractions and algebraic fractions. We state this principle as a rule.

RULE. *The value of a fraction is not changed if both the numerator and denominator are multiplied or divided by the same nonzero quantity.*

Illustrations:

$$\frac{2}{3} = \frac{8}{12}; \quad \frac{a}{b} = \frac{ac}{bc}; \quad \frac{-10}{-5} = \frac{10}{5} = 2.$$

27. Signs of Fractions. The sign of a fraction is indicated by a plus or minus symbol in the same way that the sign of an integer is indicated. Hence there are three signs associated with a fraction, the sign of the numerator, the sign of the denominator, and the sign of the fraction itself. In operations involving a fraction it is often necessary to change the sign of the fraction. The rule of signs in multiplication and division permits the following statement.

RULE 1. *The sign of a fraction and the sign of the numerator (or the denominator) may be changed without altering the value of the fraction.*

Illustrations:

$$\frac{12}{4} = -\frac{-12}{4} = -(-3) = 3;$$

$$\frac{12}{4} = -\frac{12}{-4} = -(-3) = 3;$$

$$\frac{x-y}{2x-y} = -\frac{y-x}{2x-y} = \frac{y-x}{y-2x}.$$

The rule of signs in multiplication and division is the basis for the following statement relative to a fraction in factored form.

RULE 2. *Among the factors of the numerator and the denominator of a fraction, the signs of the terms of any two factors may be changed without altering the value of the fraction; or the signs of the terms of one factor may be changed if the sign of the fraction is changed.*

Illustrations:

$$\frac{a-3}{(1-a)(2-a)} = \frac{a-3}{(a-1)(a-2)} = -\frac{3-a}{(a-1)(a-2)};$$

$$\frac{x-4}{(3-x)(x-2)} = -\frac{x-4}{(x-3)(x-2)} = \frac{4-x}{(x-3)(x-2)}.$$

28. Reducing a Fraction to Lowest Terms. A fraction is said to be reduced to its lowest terms if the numerator and denominator have no common factor other than 1. Hence, by factoring it may be determined if a fraction is in its lowest terms. Any factor appearing in both the numerator and denominator can be readily removed by division.

Example 1. Reduce $\dfrac{x^2 - 5x + 6}{x^2 + x - 6}$ to its lowest terms.

Solution. Write the fraction with its numerator and denominator in factored form:

$$\frac{(x-2)(x-3)}{(x-2)(x+3)} = \frac{x-3}{x+3}.$$

The common factor $x - 2$ is removed by dividing both the numerator and denominator by $x - 2$.

Example 2. Reduce $\dfrac{x^2 - 9x + 20}{25 - x^2}$ to lowest terms.

Solution. Factoring gives

$$\frac{x^2 - 9x + 20}{25 - x^2} = \frac{(x - 5)(x - 4)}{(5 - x)(5 + x)}.$$

The factor $x - 5$ in the numerator and the factor $5 - x$ in the denominator are alike except for signs; one is the negative of the other. A common factor results, however, by changing $x - 5$ to $5 - x$ and also changing the signs of the terms of the factor $x - 4$. Thus we have

$$\frac{(x - 5)(x - 4)}{(5 - x)(5 + x)} = \frac{(5 - x)(4 - x)}{(5 - x)(5 + x)} = \frac{4 - x}{5 + x}.$$

The intermediate steps are written here to make the process clear. The work may be shortened by noticing that a factor in the numerator and its negative in the denominator can be removed by replacing one of the factors by 1 and the other by -1.

The process of dividing a common factor from the numerator and denominator is sometimes called **canceling** the factors. Errors in this operation are often made by striking out equal quantities which are not factors. The following example points out a temptation for careless canceling.

Example 3. Reduce $\dfrac{2x + y + 5}{4x^2 - y^2}$ to lowest terms.

Solution. The numerator is prime, but factoring the denominator gives

$$\frac{2x + y + 5}{4x^2 - y^2} = \frac{2x + y + 5}{(2x + y)(2x - y)}.$$

Now $2x + y$ is a factor of the denominator. The $2x + y$ appearing in the numerator, however, is not a factor of the numerator. The numerator is not $2x + y$ *times* something; it is $2x + y$ *plus* something. To strike out the common quantity here would mean that the denominator is divided by $2x + y$ and that $2x + y$ is subtracted from the numerator. This serious error should be avoided. The fraction as given is in lowest terms.

Exercise 15

In each of the fractions **1–6** tell what value or values of x would make the denominator 0, and the fraction therefore meaningless.

1. $\dfrac{x - 3}{x - 4}$. **2.** $\dfrac{1}{(x - 1)(x + 2)}$. **3.** $\dfrac{x + a}{x^2 - a^2}$.

4. $\dfrac{2x + 5}{3x - 1}$. **5.** $\dfrac{4}{5x + 4}$. **6.** $\dfrac{1}{(3x - 2)(2x - 3)}$.

In each of the fractions **7–15** tell why like quantities may or may not be struck out (canceled) to reduce the fraction to lower terms.

7. $\dfrac{x+3}{y+3}$.
8. $\dfrac{x+y}{x}$.
9. $\dfrac{x(a+2)}{x(b-1)}$.
10. $\dfrac{(x-y)+3}{2(x-y)}$.
11. $\dfrac{(a+3)+b}{(a+3)+c}$.
12. $\dfrac{8(a+b)}{8(a+b)}$.
13. $\dfrac{3(x+1)-y}{2(x+1)+z}$.
14. $\dfrac{(x-1)(x-3)}{(x-3)(x+4)}$.
15. $\dfrac{ax+bx}{cx}$.

Reduce to lowest terms:

16. $\dfrac{25xy^2}{75x^2y}$.
17. $\dfrac{24ab^2c}{18a^2bc^2}$.
18. $\dfrac{3^2a^6b}{18a^4b^6}$.
19. $\dfrac{2x}{x^2+2x}$.
20. $\dfrac{ax+3x}{a^2+3a}$.
21. $\dfrac{3x+2}{9x^2-4}$.
22. $\dfrac{(x-2)^2}{x^2-4}$.
23. $\dfrac{a^2-ab}{ab-b^2}$.
24. $\dfrac{x-1}{x-x^2}$.
25. $\dfrac{6x-6y}{3x-3y}$.
26. $\dfrac{b^2-a^2}{(a-b)^2}$.
27. $\dfrac{3x^2+15xy}{2x^2+10xy}$.
28. $\dfrac{6abc-18ab}{3a^2bc-9a^2b}$.
29. $\dfrac{5x^3y-5x^2y^2}{x^2y^2-xy^3}$.
30. $\dfrac{a^2-b^2}{2a^2+ab-b^2}$.
31. $\dfrac{3x^2-11x+6}{3x^2+x-2}$.
32. $\dfrac{4x^2+7x-2}{1-16x^2}$.
33. $\dfrac{3+2x-x^2}{3+5x+2x^2}$.
34. $\dfrac{9-6x+x^2}{9-9x+2x^2}$.
35. $\dfrac{a^2-ab+b^2}{a^3+b^3}$.
36. $\dfrac{a^2-2ab+b^2}{a^3-b^3}$.
37. $\dfrac{x^3-y^3}{x^3+x^2y+xy^2}$.
38. $\dfrac{x^2-y^2+x+y}{x^2-y^2}$.
39. $\dfrac{4a^2-b^2}{4a^2-b^2+2a-b}$.
40. $\dfrac{y^3-y^2+y-1}{y^3+y^2+y+1}$.
41. $\dfrac{a^3-2a^2-a+2}{a^3+2a^2-a-2}$.
42. $\dfrac{c^3+3c^2+2c+6}{c^3-c^2+2c-2}$.
43. $\dfrac{8c^3+12c^2+6c+1}{8c^3+1}$.
44. $\dfrac{27c^3-1}{27c^3-27c^2+9c-1}$.
45. $\dfrac{4b^2-a^2}{a^2-4ab+4b^2+a-2b}$.
46. $\dfrac{2x^3-3x^2y+2x-3y}{3x^3+2x^2y+3x+2y}$.
47. $\dfrac{4x^2y^2+4y^2-x^2-x^4}{x^4-16y^4}$.
48. $\dfrac{3a^2b^2-b^4-3a^2+b^2}{3a^2b^2-b^4+3a^2-b^2}$.
49. $\dfrac{4a^2-12ab+9b^2-16c^2}{2a^2c-3abc+4ac^2}$.
50. $\dfrac{9x^2-y^2+4yz-4z^2}{3x^2yz+xy^2z-2xyz^2}$.

29. Multiplication and Division of Fractions. Algebraic fractions are multiplied and divided in accordance with the following rules. These rules are first met in arithmetic.

RULE 1. *The product of two or more fractions is equal to the product of the numerators divided by the product of the denominators.*

RULE 2. *To divide one fraction by another, invert the divisor and then multiply.*

The rule for the division of fractions is a direct consequence of the definition of division. That is, the quotient times the divisor is equal to the dividend. Hence,

$$\frac{a}{b} \div \frac{c}{d} = x \quad \text{if} \quad \left(\frac{c}{d}\right) x = \frac{a}{b}.$$

To obtain the quotient x, we multiply both members of the latter equation by $\frac{d}{c}$ and have

$$x = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

This result is the product of the dividend and the inverted divisor.

It is usually desirable to reduce the product, or the quotient, of fractions to lowest terms. That being the case, the best procedure is to write each fraction in factored form. Common factors of the numerator and the denominator can then be easily divided out.

NOTE. An expression which is not a fraction can be written as a fraction by using 1 for the denominator.

Example 1. Multiply $\dfrac{2x^2 - x - 3}{x^2 - 1}$ by $\dfrac{x^2 - 2x + 1}{3x^2 - x - 2}$.

Solution. Factoring each numerator and denominator of the given fractions, we have

$$\begin{aligned} \frac{(2x-3)(x+1)}{(x+1)(x-1)} \cdot \frac{(x-1)(x-1)}{(x-1)(3x+2)} &= \frac{(2x-3)(x+1)(x-1)(x-1)}{(x+1)(x-1)(x-1)(3x+2)} \\ &= \frac{2x-3}{3x+2}. \end{aligned}$$

The work may be shortened somewhat by marking through the common factors, thus:

$$\frac{(2x-3)\cancel{(x+1)}}{\cancel{(x+1)}\cancel{(x-1)}} \cdot \frac{\cancel{(x-1)}\cancel{(x-1)}}{\cancel{(x-1)}(3x+2)} = \frac{2x-3}{3x+2}.$$

Example 2. Find the indicated product:

$$(x^2 - y^2) \cdot \frac{x^2 + 3xy + 2y^2}{2x^2 - 3xy + y^2} \cdot \frac{2x - y}{x^2 + 2xy + y^2}.$$

Solution. Writing $x^2 - y^2$ in fractional form and reducing to lowest terms, we obtain

$$\frac{\cancel{(x+y)}\cancel{(x-y)}}{1}\cdot\frac{(x+2y)\cancel{(x+y)}}{\cancel{(2x-y)}\cancel{(x-y)}}\cdot\frac{\cancel{2x-y}}{\cancel{(x+y)^2}} = x+2y.$$

Example 3. Divide $\dfrac{a^3-b^3}{a^2-3ab+2b^2}$ by $\dfrac{a^2+ab+b^2}{a^2-9b^2}$.

Solution.

$$\begin{aligned}\frac{a^3-b^3}{a^2-3ab+2b^2}\div\frac{a^2+ab+b^2}{a^2-9b^2} &= \frac{a^3-b^3}{a^2-3ab+2b^2}\cdot\frac{a^2-9b^2}{a^2+ab+b^2}\\ &= \frac{(a-b)(a^2+ab+b^2)}{(a-b)(a-2b)}\cdot\frac{(a+3b)(a-3b)}{a^2+ab+b^2}\\ &= \frac{a^2-9b^2}{a-2b}.\end{aligned}$$

Example 4. Perform the indicated operations:

$$\frac{x^2-7x+12}{x^2-6x+8}\cdot\frac{x^2+3x-10}{x^2-10x+21}\div\frac{x+5}{x-7}.$$

Solution. Factoring and inverting the divisor, we have

$$\frac{\cancel{(x-3)}\cancel{(x-4)}}{\cancel{(x-2)}\cancel{(x-4)}}\cdot\frac{\cancel{(x+5)}\cancel{(x-2)}}{\cancel{(x-7)}\cancel{(x-3)}}\cdot\frac{\cancel{x-7}}{\cancel{x+5}} = \frac{1}{1} = 1.$$

Exercise 16

Perform the indicated operations, giving the results in lowest terms, and also in factored form, except for the numerical factors:

1. $\frac{3}{5}\cdot\frac{10}{42}\cdot\frac{7}{2}$.

2. $5\cdot\frac{3}{13}\cdot\frac{2}{3}$.

3. $\frac{5}{3}\cdot\frac{9}{10}\div\frac{7}{2}$.

4. $\dfrac{5x}{7y}\cdot\dfrac{28y}{15x}$.

5. $8ab^2\cdot\dfrac{3}{4ab}$.

6. $\dfrac{4x^2}{3ab}\cdot\dfrac{9a^2b}{2x}$.

7. $\dfrac{x}{3}\div 4$.

8. $x\div\dfrac{x}{4}$.

9. $64x^3y\div\dfrac{8xy^3}{3}$.

10. $\dfrac{9ab^2}{34x}\cdot\dfrac{17x^2}{27ab}$.

11. $\dfrac{5a^4x^3}{8b^2y}\cdot\dfrac{4by}{a^3x^4}$.

12. $\dfrac{64x^2}{39y}\cdot\dfrac{13y^2}{48x^3}$.

13. $\dfrac{7ac}{9b^2}\div\dfrac{14c}{a^2}$.

14. $\dfrac{10xy^3}{3z^2}\div\dfrac{5x^2y}{6z}$.

15. $\dfrac{81x^3y}{2}\div(9xy^3)$.

16. $\dfrac{x^2+x}{y^2-1}\cdot\dfrac{y+1}{x+1}$.

17. $\dfrac{a-b}{x-2y}\cdot\dfrac{2y-x}{2b-2a}$.

18. $\dfrac{x-y}{a-b}\div\dfrac{y-x}{2(a-b)}$.

19. $\frac{a^3+b^3}{x^3+y^3}\cdot\frac{x+y}{a+b}.$ 20. $\frac{1-a}{1-x^3}\cdot\frac{1-x^2}{a^2-1}.$ 21. $\frac{4-x^2}{8-x^3}\div\frac{2+x}{2-x}.$

22. $\frac{x^2-x-12}{x^2+7x-30}\cdot\frac{30-7x-x^2}{24-2x-x^2}.$ 23. $\frac{x^2-9y^2}{7x+21y}\cdot\frac{x^2-2xy-3y^2}{x^2-6xy+9y^2}.$

24. $\frac{x^2-49}{6x^2-7x+2}\cdot\frac{2bx-b}{2ax-14a}.$ 25. $\frac{2a-1}{a^3+8}\cdot\frac{a^2-4}{2a^2-5a+2}.$

26. $\frac{x^2-x-6}{x^2-x-2}\div\frac{x^2-6x+9}{x^2+8x+7}.$ 27. $\frac{2xy+2y^2}{3y^2-xy-4x^2}\div\frac{3x^2+3xy}{9y^2-16x^2}.$

28. $\frac{6x^2+7x-3}{5x^2+19x-4}\cdot\frac{x^2+8x+16}{3x^2+11x-4}.$

29. $\frac{x^3-x^2-x+1}{x^3+x^2+x+1}\div\frac{x^2-2x+1}{x^3-2x^2+x-2}.$

30. $\frac{a^3+2a^2-a-2}{a^2+3a+9}\div\frac{a^2+3a+2}{a^3-27}.$ 31. $\frac{x^4+x^2y^2+y^4}{4x^2-y^2}\cdot\frac{2xy-y^2}{x^2+xy+y^2}.$

32. $\frac{c^2-3cd+2d^2}{cx-cy+2dx-2dy}\cdot\frac{y^2-x^2}{-cx-cy+dx+dy}.$

33. $\frac{4x^2-1}{2x^2-5x-3}\cdot\frac{2x^2+3x-9}{x^2+7x+12}\cdot\frac{x^2+x-12}{4x^2-8x+3}.$

34. $\frac{x^2-6xy-7y^2}{x^2-xy-12y^2}\cdot\frac{x^2+4xy-12y^2}{x^2+7xy+6y^2}\div\frac{x^2-9xy+14y^2}{x^2-xy-12y^2}.$

35. $\frac{3a^2-a-10}{8a^2-2a-3}\cdot\frac{10a^2+a-2}{3a^2+20a+25}\div\frac{5a^2+8a-4}{12a^2+11a-15}.$

36. $\frac{x^3-9x^2+27x-27}{6x^2-19x+3}\cdot\frac{6x^2+35x+36}{x^2-6x+9}\cdot\frac{x+1}{x^2+37x+36}.$

37. $\frac{a^2+4a+4-b^2}{a^2-4a+4-b^2}\cdot\frac{a^2-2ab+b^2-4}{a^2+2ab+b^2-4}\div\frac{a^2+b^2+4-2ab+4a-4b}{a^2+b^2+4+2ab-4a-4b}.$

38. $\frac{8m^3-36n^2+54m-27}{m^2+n^2+2mn+2m+2n+1}\cdot\frac{m^2+n^2+2mn-1}{4m^3-12m^2+9m}\cdot\frac{m^2n-mn^2-mn}{m^2-n^2-2m+1}.$

39. $\frac{2x^3-4x^2-3x+6}{2x^3-3x^2+2x-3}\cdot\frac{6x^3-9x^2+2x-3}{3x^3-6x^2+x-2}\cdot\frac{x^2+1}{2x^2-3}.$

40. $\frac{x^3+y^3-x-y}{x^3+3x^2y+3xy^2+y^3}\cdot\frac{x^3+x^2y-xy^2-y^3}{x^2-xy+y^2-1}\cdot\frac{x^3-y^3+2x-2y}{x^3-x^2y-xy^2+y^3}.$

30. Addition and Subtraction of Fractions. The process of adding and subtracting algebraic fractions is just like that used in arithmetic. In algebra, however, the process is frequently called addition, even though a minus sign precedes some of the fractions. To add fractions which have different denominators, it is first necessary to express them with a common denominator. The common denominator should be the L.C.M. of the various denominators. It is then called the **lowest common denominator (L.C.D.).**

The following steps will yield the sum of two or more fractions.

1. Factor each denominator and determine the L.C.D.

2. Multiply the numerator and the denominator of each fraction by the quotient obtained by dividing the L.C.D. by the denominator of that fraction.

3. Add the resulting numerators, being careful first to reverse the signs of all terms of the numerator of any fraction preceded by a minus sign. This gives the numerator of the sum of the fractions; the denominator is the common denominator.

Example 1. Express $\frac{2}{x+3}$, $\frac{3}{x-4}$, and $\frac{5}{x+6}$ in terms of their common denominator.

Solution. The L.C.D. is the L.C.M. of the denominators, and for these fractions is $(x+3)(x-4)(x+6)$. The quotient of the L.C.D. and $x+3$ is $(x-4)(x+6)$. Hence we multiply the numerator and denominator of the first fraction by $(x-4)(x+6)$. The fraction is thus changed into an equivalent fraction having the desired denominator. Similarly, the expressions $(x+3)(x+6)$ and $(x+3)(x-4)$ are used with the other fractions. Performing these operations, we have

$$\frac{2}{x+3} = \frac{2(x-4)(x+6)}{(x+3)(x-4)(x+6)},$$

$$\frac{3}{x-4} = \frac{3(x+3)(x+6)}{(x+3)(x-4)(x+6)},$$

$$\frac{5}{x+6} = \frac{5(x+3)(x-4)}{(x+3)(x-4)(x+6)}.$$

Example 2. Add $x+1+\frac{2x-3}{x}-\frac{2x-1}{x-1}$.

Solution. The expression $x+1$ may be written as a fraction by using 1 for its denominator. The L.C.D. is $x(x-1)$. Hence,

$$\begin{aligned}\frac{x+1}{1}+\frac{2x-3}{x}-\frac{2x-1}{x-1} \\ &= \frac{(x+1)(x)(x-1)}{x(x-1)}+\frac{(2x-3)(x-1)}{x(x-1)}-\frac{(2x-1)(x)}{x(x-1)} \\ &= \frac{x^3-x+2x^2-5x+3-2x^2+x}{x(x-1)} \\ &= \frac{x^3-5x+3}{x(x-1)}.\end{aligned}$$

It should be noticed that the fraction with the numerator $(2x - 1)(x) = 2x^2 - x$ is preceded by a negative sign. This numerator, when combined with the other numerators, becomes $-2x^2 + x$.

Example 3. Add $\dfrac{2}{(x-3)(x+4)} + \dfrac{3}{(3-x)(2x+1)}$.

Solution. The factor $3 - x$ in the second denominator is the negative of the factor $x - 3$ in the first denominator. To handle this situation, we reverse the signs of the terms of $3 - x$ and change the sign before the fraction from plus to minus. Hence we obtain

$$\frac{2}{(x-3)(x+4)} - \frac{3}{(x-3)(2x+1)} = \frac{2(2x+1) - 3(x+4)}{(x-3)(x+4)(2x+1)}$$

$$= \frac{4x + 2 - 3x - 12}{(x-3)(x+4)(2x+1)}$$

$$= \frac{x - 10}{(x-3)(x+4)(2x+1)}.$$

Example 4. Add $\dfrac{x-14}{3x^2 - 7x - 6} + \dfrac{2x-4}{3x^2 + 8x + 4} + \dfrac{x+5}{x^2 - x - 6}$.

Solution. We first factor the denominators in order to find the L.C.D.

$$\frac{x-14}{(3x+2)(x-3)} + \frac{2x-4}{(3x+2)(x+2)} + \frac{x+5}{(x+2)(x-3)}$$

$$= \frac{(x-14)(x+2) + (2x-4)(x-3) + (x+5)(3x+2)}{(3x+2)(x-3)(x+2)}$$

$$= \frac{x^2 - 12x - 28 + 2x^2 - 10x + 12 + 3x^2 + 17x + 10}{(3x+2)(x-3)(x+2)}$$

$$= \frac{6x^2 - 5x - 6}{(3x+2)(x-3)(x+2)} = \frac{(3x+2)(2x-3)}{(3x+2)(x-3)(x+2)}$$

$$= \frac{2x - 3}{(x-3)(x+2)}.$$

Exercise 17

Express the fractions in each problem **1–10** as equivalent fractions having the L.C.D.

1. $\frac{1}{4x}, \frac{3}{5x^2}$.

2. $\frac{5}{3xy}, \frac{3}{2xy}$.

3. $\frac{6}{7ab}, \frac{4}{5a^2b^3}$.

4. $\frac{1}{x-1}, \frac{2}{1-x}$.

5. $\frac{4}{(x-2)^2}, \frac{5}{4-x^2}$.

6. $\frac{x}{2x-1}, \frac{4}{1-2x}$.

7. $\frac{2}{x^2-x}, \frac{x}{x^2-1}, \frac{3}{x^2+x}$.

8. $\frac{x}{3x-1}, \frac{5}{1-3x}, \frac{2x}{9x^2-1}$.

9. $\frac{a+2}{2a+4}, \frac{a-2}{3a+6}, \frac{1}{a^2-4}$.

10. $\frac{4}{(x-1)^2}, \frac{1}{(1-x)^3}, \frac{2}{x^2-1}$.

Combine into a single fraction in lowest terms:

11. $\frac{2x^2-3x+4}{3} - \frac{x^2-5x+7}{3}$.

12. $\frac{7+x+3y}{x-6} - \frac{6-5x+2y}{x-6}$.

13. $\frac{3x^2-2x+7}{(x+3)(x-3)} - \frac{3x^2-3x+4}{(x+3)(x-3)}$.

14. $\frac{20+7x+x^2}{(x-6)(x+5)} - \frac{10-6x-5x^2}{(x-6)(x+5)}$.

15. $\frac{5(x+1)}{3x-4} - \frac{7(2x-3)}{3x-4} - \frac{3(1-2x)}{3x-4}$.

16. $\frac{x(6x+5)}{5x-3} - \frac{x(x-4)}{5x-3} - \frac{12x}{5x-3}$.

17. $4 - \frac{5}{8} + \frac{7}{3}$.

18. $\frac{13}{7} - \frac{3}{5} - \frac{2}{35}$.

19. $\frac{1}{6} + \frac{3}{8} - 3$.

20. $\frac{a}{5} - \frac{a}{10} + \frac{7a}{25}$.

21. $\frac{a-2}{6} + \frac{a-5}{15}$.

22. $a - \frac{a}{3} + \frac{3a-2}{8}$.

23. $\frac{4}{a-9} - \frac{3}{a+3}$.

24. $\frac{3}{a-b} + \frac{4}{a+2b}$.

25. $\frac{a}{a-5} - \frac{3}{a}$.

26. $\frac{2}{y-3} + \frac{3}{9-y^2}$.

27. $\frac{3}{y-3} - \frac{27}{9-y^2}$.

28. $1 - y + \frac{3}{1+y}$.

29. $\frac{3}{3y-5} - 2y + 3$.

30. $a + 2b - \frac{a^2}{a-3b}$.

31. $\frac{a+3}{a^2+1} + 2a - 4$.

32. $\frac{2m}{m-3} - \frac{4}{m+6} - 1$.

33. $3 - \frac{4m}{m+3} - \frac{m+2}{m-5}$.

34. $5 - \frac{2m-3}{m-1} - \frac{2}{m}$.

35. $\frac{3m-1}{m^2-4} - \frac{2m+5}{(m-2)^2}$.

36. $\frac{m^2-1}{m^3+1} + \frac{2}{m+1}$.

37. $\frac{m+3}{m^3-m} + \frac{4}{1-m}$.

38. $\frac{4}{2x-3} - \frac{3}{2x+3} + \frac{1}{x+1}$.

39. $\frac{2}{3-x} + \frac{3}{1-x} - \frac{2}{2-x}$.

40. $\frac{x+3}{x+1} + \frac{x+4}{x-1} - \frac{x-5}{x-2}$.

41. $\frac{x-4}{x-1} - \frac{x+5}{x-2} - \frac{x+3}{x+2}$.

42. $\frac{2x}{3x-1} + \frac{3x}{3x+1} - \frac{12x^2}{9x^2-1}$.

43. $\frac{x+1}{2x-3} - \frac{x-1}{2x+3} + \frac{2x^2-7x}{4x^2-9}$.

44. $\frac{1}{x-y} - \frac{1}{x+y} + \frac{2x}{y^2-x^2}$.

45. $\frac{x-1}{x^2+3x+2} - \frac{x-13}{2x^2+3x-2}$.

46. $\frac{3}{2x^2 + x - 1} + \frac{2}{6 + x - 2x^2} + \frac{1}{4x^2 + 4x - 3}.$

47. $\frac{2}{y^2 + 7y + 12} + \frac{5y + 10}{y^2 + 3y - 4} + \frac{y + 4}{y^2 + 2y - 3}.$

48. $\frac{1}{2a^2 + 3a + 1} + \frac{5a}{2a^2 - a - 3} + \frac{a + 2}{4a^2 - 4a - 3}.$

49. $\frac{2m + 7}{2m^2 + 5m - 7} - \frac{m - 2}{2m^2 + 3m - 14} - \frac{m - 1}{m^2 - 3m + 2}.$

50. $\frac{3}{x^3 + 3x^2 - x - 3} - \frac{2}{x^3 - x^2 - 3x + 3} + \frac{4}{x^3 + x^2 - 3x - 3}.$

51. $\frac{x + 1}{2x - 3} - \frac{x - 1}{2x + 3} + \frac{2x^2 - 7x}{4x^2 - 9} - \frac{3x^2}{2x^2 + 3x - 9}.$

52. $\frac{a}{a^3 + b^3 + 2a^2b + 2ab^2} + \frac{a^2 - b^2}{a^4 + a^2b^2 + b^4} - \frac{a - b}{a^3 + b^3}.$

31. Complex Fractions. A fraction which has no fraction in its numerator or denominator is a **simple fraction.** A fraction which contains a fraction in its numerator or denominator, or both, is called a **complex fraction.**

To simplify a complex fraction means to express it as an equivalent simple fraction in lowest terms. This may be accomplished by reducing the numerator and the denominator to simple fractions and then finding the quotient. The examples illustrate the process of simplification.

Example 1. Simplify:

$$\frac{a - \frac{b}{a}}{b - \frac{a}{b}}.$$

Solution. We first reduce the numerator and the denominator to simple fractions. Thus

$$\frac{a - \frac{b}{a}}{b - \frac{a}{b}} = \frac{\frac{a^2 - b}{a}}{\frac{b^2 - a}{b}} = \frac{a^2 - b}{a} \cdot \frac{b}{b^2 - a} = \frac{b(a^2 - b)}{a(b^2 - a)}.$$

Example 2. Simplify:

$$\frac{\frac{x + 1}{x - 2} - \frac{x - 1}{x + 2}}{\frac{8}{x - 2} + 4}.$$

Solution. The lowest common denominator of the two fractions in the numerator is $(x-2)(x+2)$, and of the denominator $x-2$. Hence the given fraction becomes

$$\frac{\dfrac{x+1}{x-2}-\dfrac{x-1}{x+2}}{\dfrac{8}{x-2}+4}=\frac{\dfrac{x^2+3x+2-(x^2-3x+2)}{(x-2)(x+2)}}{\dfrac{8+4x-8}{x-2}}=\frac{\dfrac{6x}{(x-2)(x+2)}}{\dfrac{4x}{x-2}}$$

$$=\frac{6x}{(x-2)(x+2)}\cdot\frac{x-2}{4x}=\frac{3}{2(x+2)}.$$

Example 3. Perform the indicated operations:

$$\left(2-\frac{1}{x-1}\right)\div\left(2x-5+\frac{5}{x+1}\right).$$

Solution. The dividend and divisor could be written as the numerator and the denominator of a complex fraction. Hence we proceed by first reducing each part to a simple fraction. This gives

$$\frac{2(x-1)-1}{x-1}\div\frac{(2x-5)(x+1)+5}{x+1}=\frac{2x-3}{x-1}\cdot\frac{x+1}{2x^2-3x}=\frac{x+1}{x(x-1)}.$$

Example 4. Simplify:

$$\frac{a+\dfrac{1}{a-1}}{1-\dfrac{1}{a+\dfrac{1}{a}}}.$$

Solution. The denominator is itself a complex fraction. Simplifying it, we get

$$1-\frac{1}{a+\dfrac{1}{a}}=1-\frac{1}{\dfrac{a^2+1}{a}}=1-\frac{a}{a^2+1}=\frac{a^2+1-a}{a^2+1}.$$

The numerator becomes

$$a+\frac{1}{a-1}=\frac{a^2-a+1}{a-1}.$$

The quotient of the simplified forms of the numerator and denominator of the given fraction yields an equivalent fraction. Thus

$$\frac{a^2 - a + 1}{a - 1} \div \frac{a^2 - a + 1}{a^2 + 1} = \frac{a^2 + 1}{a - 1}.$$

Exercise 18

Simplify the complex fractions and other expressions:

1. $\dfrac{\frac{1}{3} - \frac{1}{5}}{2 + \frac{4}{5}}.$ **2.** $\dfrac{3 - \frac{4}{3}}{4 + \frac{3}{4}}.$ **3.** $\dfrac{5}{\frac{5}{4} - \frac{3}{8}}.$

4. $\dfrac{\dfrac{1}{x} + \dfrac{2}{3}}{4x}.$ **5.** $\dfrac{4 - \dfrac{x}{5}}{5}.$ **6.** $\dfrac{5a - a^2}{\dfrac{5}{a} - 1}.$

7. $\dfrac{\dfrac{1}{x} - 1}{\dfrac{1}{x} + 1}.$ **8.** $\dfrac{2x - \dfrac{2}{x}}{x + 1 - \dfrac{2}{x}}.$ **9.** $\dfrac{x - \dfrac{4x - 3}{x}}{2 - \dfrac{x - 1}{x^2}}.$

10. $\dfrac{x - \dfrac{x^2 - 1}{3x}}{\dfrac{4}{x} - \dfrac{2}{x^2}}.$ **11.** $\dfrac{\dfrac{1}{x} - \dfrac{2}{x^2} + 1}{\dfrac{3}{x} - \dfrac{4}{x^2} + 1}.$ **12.** $\dfrac{\dfrac{3}{x - 1} - \dfrac{1}{x + 2}}{\dfrac{x - 1}{3} - x - 2}.$

13. $\dfrac{5 - \dfrac{3}{2x - 1}}{\dfrac{1}{2x + 1} - \dfrac{x + 1}{4x^2 - 1}}.$ **14.** $\dfrac{x + 3 - \dfrac{6}{x + 2}}{x + 6 + \dfrac{7}{x - 2}}.$ **15.** $\dfrac{x + 4 - \dfrac{x + 4}{x + 6}}{x - 3 + \dfrac{x + 11}{x + 5}}.$

16. $\dfrac{\dfrac{2}{x + a} + \dfrac{3}{x - a}}{\dfrac{2}{x - a} - \dfrac{3}{x + a}}.$ **17.** $\dfrac{\dfrac{5}{9m^2 - 4} - 3}{\dfrac{3}{3m + 2} - 5}.$ **18.** $\dfrac{\dfrac{x - 3}{x + 2} - \dfrac{x + 3}{x - 2}}{\dfrac{x + 1}{x - 2} - \dfrac{x + 4}{x + 3}}.$

19. $\dfrac{\dfrac{x - y}{x + y} + \dfrac{1}{x^2 - y^2}}{\dfrac{x + y}{x - y} - \dfrac{x - y}{x + y}}.$ **20.** $\dfrac{3 - \dfrac{x - 6}{x^2 - 6x + 8}}{2 + \dfrac{x + 7}{x^2 - 2x - 8}}.$ **21.** $\dfrac{\dfrac{x - 1}{(x + 1)^2} + \dfrac{x + 1}{(x - 1)^2}}{\dfrac{x^2 + 3}{x^2 - 1}}.$

22. $\left(x - 2 - \dfrac{3}{x}\right) \div \left(x + 2 + \dfrac{1}{x}\right).$ **23.** $\left(1 - \dfrac{3}{3x^2 + 4}\right)\left(x + \dfrac{4}{3x}\right).$

24. $\left(a + \dfrac{1}{a + 1}\right) \div \left(a^2 - \dfrac{1}{a}\right).$ **25.** $\left(2 - \dfrac{5a - 10}{a^2 - 4}\right)\left(a^2 + \dfrac{a^2 + 16}{2a - 1}\right).$

26. $\left(3x + \dfrac{1}{x} - 4\right) \div \left(5x - \dfrac{2}{x} - 3\right).$ **27.** $\left(\dfrac{1}{x^2} - \dfrac{1}{y^2}\right) \div \left(x - y + \dfrac{y^2}{x + y}\right).$

28. $\left(\dfrac{x^2}{x^2 - y^2} - 1\right) \div \left(y + \dfrac{xy}{y - x}\right).$

29. $\left(m+1+\dfrac{2}{m-1}\right)\div\left(m-1-\dfrac{2}{m+1}\right).$

30. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}.$

31. $\dfrac{\dfrac{2}{1+\dfrac{1}{x}}}{\dfrac{3}{1-\dfrac{1}{x}}}.$

32. $1-\dfrac{3}{4-\dfrac{2}{1-\dfrac{1}{x}}}.$

33. $3-\dfrac{4}{2+\dfrac{5}{3+\frac{1}{4}}}.$

34. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x-1}}}.$

35. $\dfrac{x}{1-\dfrac{1}{x+\dfrac{1}{x+1}}}.$

Chapter Four

EQUATIONS

32. Equations. An **equation** is a statement which says that two quantities are equal. The usual way of writing an equation is to place the symbol = (read "is equal to") between the two equal quantities. An equation then has two **sides,** or **members:** the left-hand or left member and the right-hand or right member. Equations usually have one or more letters which are regarded as the **unknowns.** The unknowns are usually indicated by letters chosen from the latter part of the alphabet. A set of numbers which, when substituted for the unknowns, makes the two members of an equation equal is said to **satisfy** the equation; and the set of numbers is called a **solution** of the equation. A solution of an equation having one unknown is called a **root** of the equation.

Illustrations:

$$\text{(a) } x - 3 = 1, \quad \text{(b) } x^2 - 4x + 4 = (x - 2)^2.$$

In equation (a) the unknown x can obviously stand only for the number 4; that is, 4 is the root or solution of the equation. But equation (b) will be satisfied by any number whatsoever for x. To observe that this is the case, square $x - 2$ as indicated and write the equation in the form

$$x^2 - 4x + 4 = x^2 - 4x + 4,$$

where the two members are exactly alike. Hence, we have illustrations of two different kinds of equations.

An equation which is satisfied by replacing each of the unknowns by any number whatsoever, for which its members have meaning, is called an **identity.** An equation which is not an identity is called a **conditional equation.** A conditional equation places restrictions on the unknowns, and it may be regarded as an inquiry as to what are the solutions.

If it is desired to specify that an equation is an identity, the symbol $\equiv$ (read "is identically equal to") is used in place of the equal sign. For example,

$$\frac{2x}{x^2 - 1} \equiv \frac{1}{x - 1} + \frac{1}{x + 1},$$

as may be seen by adding the two fractions in the right-hand member. It should be noticed that this identity would have no meaning for $x = 1$, or $x = -1$, since either of these values would produce a denominator which is zero. But all other numbers are permissible for x.

A problem of much interest in algebra is that of finding the solutions of equations. In this chapter we introduce the problem by considering equations which have one unknown and which can be expressed so that only the first power of the unknown is involved.

33. Operations on Equations. Two equations are said to be *equivalent* if their solutions are exactly the same. The equations $x - 3 = 2$ and $x + 1 = 6$, for example, are equivalent; each has the one root 5. Neither of these equations, however, is equivalent to $(x - 5)(x - 1) = 0$. The values $x = 5$ and $x = 1$ satisfy the last equation. But $x = 1$ does not satisfy the other equations.

In the process of finding the solutions of an equation it is usually necessary to perform operations which lead to other equations. In this connection it is important to know whether the derived equations are equivalent to the original equation. Pertaining to this question we state rules for the fundamental operations.

RULE 1. *If the same quantity is added to, or subtracted from, both members of an equation, an equivalent equation is obtained.*

RULE 2. *If both members of an equation are multiplied by, or divided by, the same nonzero quantity which does not contain any unknowns, an equivalent equation is obtained.*

RULE 3. *If both members of an equation are multiplied or divided by a quantity which contains an unknown, the equation obtained will in some cases not be equivalent to the original one.*

Illustrations: Adding 8 to both members of $4x - 8 = 12$, we get $4x = 20$. It is easy to see that 5 is a root of each of these equations, and that no other value for x will satisfy either equation. Hence the two equations are equivalent.

If we divide both sides of $4x - 8 = 12$ by 4, the equation $x - 2 = 3$ is obtained. These equations are equivalent because each has the one root 5.

Multiplying both members of $4x - 8 = 12$ by x gives the equation $4x^2 - 8x = 12x$. The derived equation is satisfied by $x = 5$ and also by $x = 0$; whereas the first equation is not satisfied by $x = 0$. Conversely, if we start with $4x^2 - 8x = 12x$ and divide by x, a root is lost rather than gained. Notice, however, that each of these operations, yielding nonequivalent equations, involves the unknown.

A root of a derived equation which is not a root of the original equation is called an **extraneous** root. An extraneous root can be detected by substituting in the given equation. An operation which leads to an equation with fewer solutions than the given equation should be avoided if all the roots are desired.

34. Solving Linear Equations in One Unknown. To **solve** an equation means to find all the numbers which may be substituted for its unknowns to satisfy the equation. The simplest problem of this kind is that of solving an equation of the form

$$ax + b = 0,$$

where b stands for any number and a may be any number except zero. This equation is of the first degree in x and is called a **linear equation.** The unknown x may be found in the following way.

$$ax = -b, \quad (b \text{ is subtracted from both sides})$$

$$x = \frac{-b}{a}, \quad (\text{both sides are divided by } a)$$

The solution, $-b/a$, could be used as a formula for solving equations of the form $ax + b = 0$. But it is better to apply the method here used to solve each particular equation. It should be noticed that each step used in the process of solving the equation involves an operation which yields an equivalent equation.

Example 1. Solve $3x - 5 = -x + 7$.

Solution. Adding 5 to both sides, we obtain

$$3x = -x + 12.$$

Adding x to both sides gives

$$4x = 12.$$

Both sides are then divided by 4, and

$$x = 3.$$

Check: Replacing x in the given equation by 3, we obtain in the left member $3(3) - 5 = 9 - 5 = 4$; and the right side becomes $-3 + 7 = 4$.

Adding 5 to both sides of $3x - 5 = -x + 7$ yielded the equivalent equation $3x = -x + 12$. It may be noticed that the same result is obtained if we take the term -5 from the left-hand side and write $+5$ as a term of the right-hand side. And, similarly, the $-x$ can be brought to the left-hand side if we but change its sign. The process of taking a term from one side of an equation to the other and changing its sign is called **transposing the term.** If the student employs this process, he should understand that in effect the same quantity is added to or subtracted from both members of the equation.

Example 2. Solve $\frac{2x}{3} - \frac{5x}{6} = \frac{2}{3} - \frac{17}{6}$.

Solution. To get rid of the fractions, multiply both sides of the equation by 6, the L.C.M. of the denominators.

$$\frac{6(2x)}{3} - \frac{6(5x)}{6} = \frac{6(2)}{3} - \frac{6(17)}{6}.$$

Reducing each fraction, we have

$$4x - 5x = 4 - 17.$$

Collecting terms gives

$$-x = -13.$$

Both sides are then multiplied by -1, and

$$x = 13.$$

Check: To check this result, we substitute 13 for x in the given equation.

$$\frac{2(13)}{3} - \frac{5(13)}{6} = \frac{2}{3} - \frac{17}{6}$$

$$\frac{52 - 65}{6} = \frac{4 - 17}{6}$$

$$\frac{-13}{6} = \frac{-13}{6}.$$

Example 3. Solve $3x + 4 = 2ax + 5c$ for x.

Solution. To solve this equation, we first transpose terms so that all those involving x are on the left side and the other terms are on the right. This gives

$$3x - 2ax = 5c - 4.$$

Factoring, we obtain

$$x(3 - 2a) = 5c - 4,$$

and dividing by $3 - 2a$ gives

$$x = \frac{5c - 4}{3 - 2a}.$$

The reader should check this result by substituting it for x in the given equation.

Exercise 19

Show that equations **1–6** are identities:

1. $3(x - 4) = 3x - 12.$
2. $4(b + c) = 4b + 4c.$
3. $x^2 + 4x + 4 = (x + 2)^2.$
4. $(x + y)(x - y) = x^2 - y^2.$
5. $\dfrac{x^2 - x + 1}{x - 1} = x + \dfrac{1}{x - 1}.$
6. $\dfrac{3x + 1}{x^2 + x} = \dfrac{1}{x} + \dfrac{2}{x + 1}.$

Solve for x, and check by substitution:

7. $8x - 15 = 3x.$
8. $5x - 4 = 3x - 6.$
9. $x - 5 = 5x + 11.$
10. $x - 4x = 5 + 2x.$
11. $3 + 7 = 5x - 3x.$
12. $2x - 9 = 6x + 5x.$
13. $\frac{1}{2}x - \frac{1}{6}x = \frac{1}{3}.$
14. $\frac{1}{3}x + \frac{1}{4}x = \frac{7}{6}.$
15. $\frac{2}{3}x - \frac{1}{2}x = \frac{1}{3}.$
16. $\frac{2}{5}x + \frac{1}{3}x = -\frac{22}{15}.$
17. $\frac{3}{2}x + \frac{1}{3} = \frac{1}{5}x + \frac{41}{6}.$
18. $\frac{5}{2}x - \frac{2}{3}x + \frac{55}{6} = 0.$
19. $\frac{3}{4}x = \frac{25}{2} - \frac{4}{3}x.$
20. $\frac{1}{6}x - x + \frac{2}{3} - \frac{3}{2} = 0.$
21. $x + \frac{2}{3}x = \frac{35}{3}.$

22. $3(4 - 5x) - 2(x + 4) = 4.$
23. $2(x - 7) = -3(2x + 1) - 3.$
24. $-9 = 7(x - 1) - 3(2x + 3).$
25. $3(7 - x) - 5(3 + 2x) - 6 = 0.$
26. $9 = 5x - 2 + 3(2x + 1).$
27. $4x + 13 = 3(5x + 20) + 8.$
28. $0.2x - 4.3 = -1.1x + 0.9.$
29. $1.2x + 3.4x + 5.8 = -3.4.$
30. $1.2x - 0.1x + 0.8x = -9.5.$
31. $3.0x - 7.1x = 2.6x - 5.9.$
32. $x + 4 - x^2 = x(2 - x).$
33. $(3x - 1)(x + 1) = 3x^2.$
34. $(x + 1)^2 - (x - 1)^2 = x + 9.$
35. $(x - 4)^2 = (x + 5)^2.$
36. $(2x - 3)^2 + 4x(4 - x) = 5.$
37. $(2x - 3)(3x + 4) = 6x^2.$

Solve for y:

38. $\dfrac{3y - 4}{6} + \dfrac{5 + 2y}{3} = \dfrac{1}{18}.$
39. $\dfrac{4y + 1}{3} - \dfrac{3y + 2}{5} = \dfrac{2}{15}.$
40. $\dfrac{2 - 4y}{5} - \dfrac{2y - 3}{2} = -\dfrac{3}{2}.$
41. $\dfrac{3y + 3}{7} - \dfrac{2y + 1}{3} = \dfrac{5}{21}.$
42. $\dfrac{5y + 4}{4} + \dfrac{5}{12} = \dfrac{1 - 2y}{3}.$
43. $\dfrac{y + 2}{2} - \dfrac{y + 3}{3} = \dfrac{2y + 3}{4}.$
44. $\dfrac{1 - y}{2} = \dfrac{1 - 2y}{4} + \dfrac{1 - 3y}{6}.$
45. $\dfrac{3 - 4y}{5} = \dfrac{4 - 3y}{4} - \dfrac{5 - 2y}{2}.$
46. $\dfrac{2y + 7}{4} + \dfrac{y - 5}{3} = \dfrac{y + 1}{6}.$
47. $\dfrac{y}{7} + \dfrac{y - 1}{3} = \dfrac{2y + 1}{21} - \dfrac{1}{3}.$

Solve the following equations for x:

48. $2x + 3a = 5 + 4a + x$.
49. $3x - 4b = x + a - 2y$.
50. $5x - 7 - 2a = -x + 3a$.
51. $a - 2x = 3x + 2b + 4$.
52. $6x - 5a + 6 = 5 + 5x$.
53. $10 - 4a - x = 3x + b$.
54. $x(5 + 4b) + 4 = 0$.
55. $x(a - 3) + 5 = 2b - 6$.
56. $x(10 - 3a) = 4 + 2b$.
57. $x(2a - 5) - 3 + 2a = 0$.
58. $2x + 3a = ay + 4 + a$.
59. $3 - ax + 4x - 3b = 0$.
60. $2(x - a) - 5 = 3(ax - 4)$.
61. $ax - 7 = x + a + 3$.
62. $3x + 7 = cx - bx + 3a$.
63. $6 - 3ax = a - 2bx + c$.
64. $3x(5 - 3y) + 5 = 2x - y$.
65. $5x(3 + 3y) + 7 = 2y$.

66. $(x + 4)(x - 1) = (x + 2)(x - 3)$.
67. $x(x - 4) - 3x^2 = (2x - 1)(1 - x) - 7$.
68. $3x(x - 1) - (2x + 3)(x + 5) = x^2 - 2x + 1$.
69. $(ax + 2)(ax - 1) = ax(ax + 5) + 9$.
70. $(mx + 3)(mx - 3) - mx(mx - 4) = 8$.
71. $(3 + 2ax)(2 - 3ax) = 4 - 6ax(ax + 2)$.
72. $(x - 2y)(x + y) = x^2 + 3xy + 4$.
73. $(y + 2x)(y - x) + 2x^2 + 4 - 2y = 0$.
74. $(2y - x)(y + 3x) = 4xy - 3x^2 + 5$.
75. $(x + y + 1)^2 - (x - y + 1)^2 = x + y + 2$.
76. $(x - y - 2)^2 + x - y - 2 = (x + y)^2 + 4$.

35. Equations with the Unknown in the Denominator. An equation having simple fractions may be written in non-fractional form by multiplying both members by an expression which is divisible by each of the denominators. This process is called **clearing the equation of fractions.** If none of the denominators contains the unknown, the resulting equation is equivalent to the original one. If, on the other hand, the unknown appears in any denominator, the resulting equation may have roots in addition to those of the original equation. Here, however, we shall deal with fractional equations which become linear when the equation in each case is multiplied by the L.C.M. of its denominators. The resulting equation will have but one root which will also be a root of the original equation except for the case in which it makes a denominator zero.

Example 1. Solve $\dfrac{1}{x-3} + 1 = \dfrac{1}{x-3}$.

Solution. Multiplying both members by $x - 3$, we get

$$1 + x - 3 = 1,$$

and

$$x = 3.$$

But 3 is not a root of the given equation since it makes a denominator zero. In fact, the equation has no solution, since the left member exceeds the right member by 1 for any permissible value of x. Although the given equation has no solution, the new equation, obtained by clearing of fractions, does have 3 as a root.

Example 2. Solve the equation $\dfrac{1}{x+1} + \dfrac{2}{x-1} = \dfrac{x+5}{x^2-1}$.

Solution. The L.C.M. of the denominators is $(x+1)(x-1)$. Hence, multiplying gives

$$\frac{(x+1)(x-1)}{x+1} + \frac{2(x+1)(x-1)}{x-1} = \frac{(x+1)(x-1)(x+5)}{x^2-1}.$$

Then, reducing each fraction, we have

$$x - 1 + 2x + 2 = x + 5,$$

$$2x = 4,$$

$$x = 2.$$

Check:

$$\frac{1}{2+1} + \frac{2}{2-1} = \frac{2+5}{4-1},$$

$$\tfrac{1}{3} + 2 = \tfrac{7}{3},$$

$$\tfrac{7}{3} = \tfrac{7}{3}.$$

Example 3. Solve the equation $\dfrac{x-1}{2x-1} - \dfrac{x}{x+1} = \dfrac{-x^2+2}{2x^2+x-1}$.

Solution. Multiplying by $(2x-1)(x+1)$, the L.C.M. of the denominators, gives

$$(x-1)(x+1) - x(2x-1) = -x^2 + 2.$$

Then

$$x^2 - 1 - 2x^2 + x = -x^2 + 2,$$

and

$$x = 3.$$

Check:

$$\frac{3-1}{6-1} - \frac{3}{3+1} = \frac{-9+2}{18+3-1},$$

$$\frac{2}{5} - \frac{3}{4} = \frac{-7}{20},$$

$$\frac{8-15}{20} = \frac{-7}{20},$$

$$\frac{-7}{20} = \frac{-7}{20}.$$

Exercise 20

Solve each equation for x, and reject as a root any number which makes a denominator equal to zero:

1. $\frac{1}{x} - \frac{1}{4} = \frac{1}{12}$.

2. $\frac{11}{12} + \frac{3}{4} = \frac{1}{x}$.

3. $\frac{3}{2} + \frac{2}{x} = \frac{11}{5}$.

4. $\frac{2}{x} - \frac{2}{5} = \frac{1}{x} + \frac{11}{10}$.

5. $\frac{2}{3x} + \frac{1}{2} = \frac{3}{2x} + \frac{13}{6}$.

6. $\frac{5}{7x} - \frac{7}{2x} - 3 = \frac{3}{14}$.

7. $\frac{2}{x-1} + \frac{3}{x+1} = \frac{4}{x^2-1}$.

8. $\frac{1}{2x-1} - \frac{3}{2x+1} = \frac{3}{4x^2-1}$.

9. $\frac{2x+1}{x} + \frac{4}{x^2-x} = \frac{2x}{x-1}$.

10. $\frac{x}{x-2} + \frac{2}{x+2} = \frac{x^2-1}{x^2-4}$.

11. $\frac{2x-1}{x} - \frac{6}{x+3} = \frac{2x^2+1}{x^2+3x}$.

12. $\frac{x^2+1}{x^2-x-2} = \frac{3}{x-2} + 1$.

13. $\frac{4x+2}{x^2-1} - \frac{1}{x+1} = \frac{1}{x-1}$.

14. $\frac{1-5x^2}{3x^2-2x-1} = \frac{1-2x}{3x+1} - 1$.

15. $\frac{x+1}{x} - \frac{x+4}{x+5} = \frac{3x+5}{x^2+5x}$.

16. $\frac{x^2+1}{x^2+x-6} = \frac{3x+8}{x+3} - \frac{2x-5}{x-2}$.

17. $\frac{x-1}{x+1} - \frac{1}{x+4} = \frac{x^2+x-9}{x^2+5x+4}$.

18. $\frac{2}{x+a} - \frac{3}{x-a} = \frac{a}{x^2-a^2}$.

19. $\frac{a}{x-1} + \frac{2}{x+1} = \frac{a+2}{x-2}$.

20. $\frac{a+2x}{2(x+a)} + \frac{a+x}{a-x} = \frac{2a}{a+x}$.

21. $\frac{1}{2x+a} + \frac{2}{3x} = \frac{7}{6x-3a}$.

22. $\frac{a}{ax-1} = \frac{a}{ax+1} + \frac{ax+1}{a^2x^2-1}$.

23. $\frac{1}{x-4} + \frac{1}{x+2} - \frac{1}{x+1} - \frac{x-2}{x^2+3x+2} = 0$.

24. $\frac{2}{x^2-2x-15} + \frac{3}{x^2-x-20} + \frac{4x}{x^2+7x+12} = \frac{4}{x+4}$.

25. $\frac{2x+5}{2x-5} + \frac{4x-5}{x+11} - \frac{10x^2-15x+32}{4x^2+17x-55} = 0$.

26. $\frac{x+7}{(x+1)(x-2)(x+3)} + \frac{5}{(x-2)(x+3)} = \frac{2}{(x+1)(x+3)}$.

27. $\frac{x^2+2x+1}{(x^2+2x-3)(x+3)} - \frac{1}{x^2+4x+3} = \frac{x-2}{x^2+2x-3}$.

28. $\frac{x+3}{x^2-5x+4} - \frac{x-4}{x^2+2x-3} + \frac{2}{x^2-x-12} = 0$.

29. $(3x+4) \div (6x-5) = (2x+5) \div (4x-1)$.

30. $(x+6) \div (2x^2+5x) = (x+1) \div (2x^2+x-10)$.

31. $(3x+5) \div (3x^2+11x-4) = (x-2) \div (x^2+7x+12)$.

36. Stated Problems. The equations in Exercises 19 and 20 are either linear or expressible linearly by clearing of fractions. We next consider worded problems which lead to equations of the same type. The equation in each case is to be found from the described relations between known numbers and unknown numbers. Such problems, given in words, are called **stated problems.** There is a great variety of stated problems, and specific directions cannot be given for finding the necessary equations. But we make the following general suggestions:

1. Read and reread the problem until it is clear what is stated, getting well in mind the known numbers and the unknown numbers.

2. Express the unknowns in terms of a single letter.

3. Find the quantities, involving the knowns and unknowns, which are equal; and then form an equation.

4. Solve the equation, and check the result.

Example 1. A rectangle whose length exceeds its width by 2 feet has the same area as a rectangle which is 3 feet narrower and 4 feet longer. Find the dimensions of the rectangles.

Solution. Here we have four unknowns which can be readily expressed in terms of one letter.

Let $x =$ the width of the first rectangle, and

$x + 2 =$ the length of the first rectangle.

Then

$x - 3 =$ the width of the second rectangle,

and

$x + 6 =$ the length of the second rectangle.

Hence,

$x(x + 2) =$ area of first rectangle,

and

$(x - 3)(x + 6) =$ area of second rectangle.

Since the rectangles have equal areas, we can form the equation

$$x(x + 2) = (x - 3)(x + 6).$$

Whence,

$$x^2 + 2x = x^2 + 3x - 18,$$

$$-x = -18,$$

$$\left.\begin{aligned} x &= 18 \\ x + 2 &= 20 \end{aligned}\right\} \text{dimensions of first rectangle.}$$

$$\left.\begin{aligned} x - 3 &= 15 \\ x + 6 &= 24 \end{aligned}\right\} \text{dimensions of second rectangle.}$$

Check:

$$(18)(20) = 360.$$

$$(15)(24) = 360.$$

Example 2. How many ounces of pure silver must be added to 100 ounces, 40% pure, to make a mixture which is 65% pure?

Solution. The equation may be set up by noticing that the amount of silver in the two separate parts is just the same as the amount of silver in the mixture. Using x to stand for the number of ounces of silver to be added, we arrange the information thus:

x oz.	and	100 oz.	give	$(100 + x)$ oz.
100% pure		40% pure		65% pure
x oz. silver		40 oz. silver		$.65(100 + x)$ oz. silver

Since the original 40 ounces of silver and the additional x ounces constitute the silver in the mixture, we have

$$x + 40 = .65(100 + x)$$

$$.35x = 25,$$

$$x = 71\tfrac{3}{7} \text{ oz.}$$

Check:

$$71\tfrac{3}{7} + 40 = 111\tfrac{3}{7} \quad \text{and} \quad .65(171\tfrac{3}{7}) = 111\tfrac{3}{7}.$$

Example 3. A can do a piece of work in 7 days, and B can do the same work in 10 days. How many days would it take for both to do the work together?

Solution. The clue to the solution of this problem may be had by noticing that the fractional part of the work done in one day is 1 divided by the number of days required for doing all the work.

Let x = the number of days required to do the work together.

Then

$$\frac{1}{x} = \text{the part of the work completed in 1 day,}$$

$$\tfrac{1}{7} = \text{the part done by A in one day,}$$

$$\tfrac{1}{10} = \text{the part done by B in one day,}$$

and

$$\tfrac{1}{7} + \tfrac{1}{10} = \text{part done together in 1 day.}$$

Hence,

$$\frac{1}{7}+\frac{1}{10}=\frac{1}{x},$$

and, clearing of fractions, we get

$$\begin{aligned} 10x + 7x &= 70, \\ 17x &= 70, \\ x &= 4\tfrac{2}{17} \text{ days.} \end{aligned}$$

Example 4. A man rows downstream at the rate of 5 miles per hour and upstream at the rate of 2 miles per hour. How far downstream should he go if he is to return in $1\frac{3}{4}$ hours after leaving?

Solution. If a body travels at a constant speed for a given time, the relation among the distance d, the speed s, and the time t is expressed by each of the formulas

$$d = st, \quad s = \frac{d}{t}, \quad t = \frac{d}{s}.$$

Hence, letting x stand for the distance in our problem, we have $x/5$ as the time used in going downstream and $x/2$ the time upstream. These times add to $1\frac{3}{4}$ and, therefore,

$$\begin{aligned} \frac{x}{5}+\frac{x}{2} &= \frac{7}{4}, \\ 4x + 10x &= 35, \\ x &= 2\tfrac{1}{2} \text{ miles.} \end{aligned}$$

Check: The time downstream is $2\frac{1}{2} \div 5 = \frac{1}{2}$ hour; the time up is $2\frac{1}{2} \div 2 = 1\frac{1}{4}$ hours. The total time is $1\frac{3}{4}$, which shows the result to be correct.

Exercise 21

1. Find two numbers one twice the other, which add to 141.

2. Find the number which increased by $\frac{3}{4}$ of itself becomes 133.

3. The sum of three consecutive integers is 156. Find the integers.

4. The sum of two numbers is 110, and one of the numbers exceeds the other by 42. Find the numbers.

5. The sum of three numbers is 171. The second number is $\frac{1}{2}$ of the first and the third number is $\frac{3}{4}$ of the first. Find the numbers.

6. The sum of three numbers is 92. The second is 3 more than the first and the first is 5 less than the third. Find the numbers.

7. Each of the two equal angles of an isosceles triangle is 15° more than the third angle. Find the angles, remembering that the sum of the angles of a triangle is 180°.

8. One angle of a triangle is twice the smallest and $\frac{2}{3}$ of the largest angle of the triangle. Find the three angles.

9. The smallest angle of a triangle is $\frac{2}{3}$ of the largest angle and 19° less than the other angle. Find the angles.

10. The width of a rectangle is the same as the side of a square, and the length is 4 units more than the width. Find the dimensions of the rectangle if its area is 52 square units more than the area of the square.

11. Two triangles have equal bases. The altitude of one triangle is 3 units more than its base and the altitude of the other is 3 units less than the base. Find the altitudes if the areas of the triangles differ by 21 square units.

12. The length of a rectangle exceeds its width by 5 units. If each dimension were increased by 2 units, the area would change by 62 square units. Find the dimensions of the rectangle.

13. A purse contains $11.65 in quarters and dimes. If the total number of coins is 70, find how many there are of each.

14. In a contest $2000 cash is offered in $20 and $5 prizes. Find the number of each kind of prize if there are 256 altogether.

15. At a football game 22,000 tickets were sold for $23,000. The admission price was $1.50 for an adult and 25 cents for a child. How many tickets of each kind were sold?

16. How many pounds of pecans worth 45 cents per pound should be mixed with 100 pounds of almonds worth 34 cents per pound to make a mixture worth 40 cents per pound?

17. How many ounces of pure silver must be added to 24 ounces, 60% pure, to make a mixture 76% pure?

18. How much milk which is 3% butterfat and how much cream which is 30% butterfat should be mixed to make 100 gallons of milk which is 4% butterfat?

19. The time for walking a certain distance at 4 miles per hour is 30 minutes longer than is required at 5 miles per hour. Find the distance.

20. To fly a certain distance at 280 miles per hour requires 30 minutes more than would be required at a speed of 320 miles per hour. Find the distance.

21. A car went from one town to another in 1 hour. The speed over half the distance was 40 miles per hour and 60 miles per hour over the other half. Find the distance.

22. A student has an average of 74 on three tests. What grade on the fourth test will bring the average to 80?

23. A student has an average of 74 in a certain course preceding the final examination. If the final examination counts 40%, what should be made on it to bring his grade to 80?

24. In college algebra a student earns an average grade of 86 on his homework and 74 on his hour tests. What final examination grade will give him a total average of 80 if homework counts $\frac{1}{5}$, hour tests $\frac{3}{5}$, and the final examination $\frac{1}{5}$?

25. A man is 46 years older than his son. In ten years he will be twice as old as his son. Find their present ages.

26. Five years ago a father was three times as old as his son. Five years from now he will be twice as old as his son. Find their present ages.

27. A son is 22 years younger than his father. In 2 years he will be $\frac{1}{3}$ as old as his father. Find their ages now.

28. The sum of two numbers is 72, and the larger divided by the smaller produces a quotient of 4 and a remainder of 2. Find the numbers.

29. The denominator of a fraction is 8 more than the numerator. If 1 is added to the numerator and 11 is subtracted from the denominator, the new fraction is equal to the reciprocal of the first fraction. Find the first fraction.

30. The denominator of a fraction is 8 more than the numerator. If 2 is added to the numerator and 6 is added to the denominator, the resulting fraction is equal to the original fraction. Find the original fraction.

31. A river flows at 3 miles per hour. Find the speed of a motorboat which goes upstream 2 miles in the same time required to go 3 miles downstream.

32. The speed of a motorboat is 12 miles per hour in still water. Find the speed of a river's current if the motorboat goes 4 miles downstream in the same time required to go 3 miles upstream.

33. A man drove 210 miles in $3\frac{1}{3}$ hours. Part of the trip was at 60 miles per hour and the rest at 65 miles per hour. Find the time spent at the lower speed.

34. A can do a piece of work in 6 days, and B can do the same work in 5 days. How long will it take to do the job if both work?

35. A piece of work can be done by A in 7 days, by B in 8 days, and by C in 10 days. How long will it take all three working together to do the job?

36. A can paint a house in 10 days, and B can paint the house in 12 days. How long will it take to paint the house if both work for 5 days, and then A finishes the work?

37. A can do a job in 10 days, and B can do the job in 12 days. After A works alone for 1 day, both work until the job is completed. How many days does B work?

38. A can do a task in 10 hours, B in 12 hours, and C in 13 hours. How long will it take to do the task if A and B work 3 hours, and then B and C complete it?

39. An alloy of silver and gold weighs 30 ounces in air and 28 ounces in water assuming that silver loses $\frac{1}{10}$ of its weight in water and that gold loses $\frac{1}{19}$ of its weight, find how many ounces of each metal are in the alloy.

40. Two men starting at a point on a 1-mile circular race track walk in opposite directions and meet in 6 minutes. But if they walk in the same direction, the faster gains one lap in 30 minutes. Find the speed of each man in miles per hour.

41. A man loans \$20,000, part at $3\frac{1}{2}\%$ annual interest and the rest at 4%. Find the amount of each investment if the yearly income is \$755.

42. A man loans \$5000 at one interest rate and \$6000 at a 1% greater rate. The \$6000 loan earns \$85 more per year than the \$5000 loan. Find the interest rates.

43. One pipe can fill a tank in 7 hours. Another pipe can empty the tank in 12 hours. How long will it take to fill the tank if both pipes are open?

44. A tank can be filled by one pipe in 18 minutes, by another in 24 minutes, and emptied by a third pipe in 12 minutes. Starting empty, how long will it take to fill the tank with the three pipes open?

45. How many quarts of a 20% solution of alcohol should be mixed with 20 quarts of a 50% solution to have a 40% solution?

46. Town A is 13 miles west of town B. A man walks from A to B at the rate of 3 miles per hour, and another man, starting at the same time, walks from B to A at the rate of 4 miles per hour. Find the times after starting when the men are 2 miles apart.

47. A car radiator has a capacity of 12 quarts. It is filled with a 30% antifreeze solution. How much should be drained out and replaced with antifreeze to bring the solution to 40% antifreeze?

Chapter Five

FUNCTIONS AND GRAPHS

37. Constants and Variables. We consider now two special kinds of quantities which are used extensively in problems of mathematics. One is a quantity which is fixed in value and is called a **constant.** The other is a quantity to which different values may be assigned and is called a **variable.** The values which a variable may take is called its **range.** Letters are used to stand for variables. Numbers, and sometimes letters, are used to denote constants. The following examples illustrate the ideas of constants and variables.

Suppose two towns are 100 miles apart and that a train travels at a uniform speed of 50 miles per hour from one town to the other. If we let t stand for the time in hours since the train passed the first town, then the miles d from the town is given by

$$d = 50t.$$

In this formula the number 50 is a constant. The letters t and d are variables. The time t starts at zero and increases to 2; the distance d varies from zero to 100.

As a further illustration, consider the formula for the weight of a ball. If r is the radius and d the density (weight per cubic unit of volume), then the weight w is given by

$$w = \tfrac{4}{3}\pi r^3 d.$$

This formula may be used to find the weight of a ball of any radius and any density. The letters r, d, and w are therefore variables. The numbers $\frac{4}{3}$, π, and the exponent 3 are constants. If, however, we apply the formula to find the weights of balls of different sizes but made of the same kind of material, then d assumes the role of a constant with r and w the variables.

38. Functions. Each of the formulas of the preceding section expresses a relation of variables. In mathematics much study is given to variables so related that one depends on another for its values. In this connection, we have the following definition.

DEFINITION. *If for each value which one variable may have the value of a second variable is determined, then the second variable is said to be a* **function** *of the first.*

In accordance with this definition, the circumference of a circle is a function of its radius. No matter what the radius of a circle is, the circumference is given by $c = 2\pi r$. The area of a circle is also a function of its radius. The volume of a cube is a function of its edge.

Usually we assign values to one of two variables to obtain the corresponding values of the other. The variable to which we assign values, or think of assigning values, is called the **independent variable.** The variable whose corresponding values are thus determined is called the **dependent variable.** If, for example, we use $c = 2\pi r$ to find circumferences of circles of given radii, then r is the independent variable and c is the dependent variable. We could, of course, reverse the situation and consider c as the independent variable with r the dependent variable.

39. Useful Notation for Functions. Suppose we say that a variable y is a function of a variable x, and that the relation is given by

$$y = x^3 - 2x^2 + 3.$$

Clearly, y is a function of x, since for each value of x the value of y can be found. Now, to indicate that y depends on x, let x be placed in parentheses to the right of y, as $y(x)$. Thus we have

$$y(x) = x^3 - 2x^2 + 3.$$

The notation $y(x)$ is read "y of x"; it should not be confused with multiplication.

Other letters than x and y, of course, may stand for variables. And in a problem where there are different functions of a variable x, we could designate them by $f(x)$, $g(x)$, $h(x)$, etc.

Suppose, in $y(x) = x^3 - 2x^2 + 3$, we wish to find the value of y when x takes the value 2. We simply substitute 2 for x and write

$$y(2) = 2^3 - 2(2)^2 + 3 = 3.$$

Hence $y(2)$ stands for the value of y when $x = 2$; and $y(2) = 3$ gives a pair of corresponding values of the variables.

Example. If $h(s) = 3s^2 - 4s + 5$, find $h(-4)$, $h(2s)$, and $h(y - 1)$.

Solution. The number or expression written in parentheses after h tells what value is to be given to the independent variable s. Hence, by substituting, we obtain

$$h(-4) = 3(-4)^2 - 4(-4) + 5 = 48 + 16 + 5 = 69;$$

$$h(2s) = 3(2s)^2 - 4(2s) + 5 = 12s^2 - 8s + 5;$$

$$h(y - 1) = 3(y - 1)^2 - 4(y - 1) + 5 = 3y^2 - 10y + 12.$$

Exercise 22

1. If $f(x) = 5x - 3$, find $f(-2), f(0), f(4), f(\frac{1}{2}), f(-\frac{3}{4})$.
2. If $f(x) = x^2 + x - 1$, find $f(0), f(1), f(-1), f(\frac{1}{2}), f(-\frac{3}{2})$.
3. If $g(x) = 2x^2 - 5x + 3$, find $g(0)$, $g(2)$, $g(-3)$, $g(\frac{1}{2})$, $g(-\frac{1}{4})$.
4. If $g(x) = 6 - x - x^2$, find $g(2)$, $g(-3)$, $g(\frac{1}{3})$, $g(-\frac{1}{3})$, $g(4)$, $g(-4)$.
5. If $y(x) = x^3 - x^2 - 2x$, find $y(0)$, $y(2)$, $y(-1)$, $y(\frac{1}{2})$, $y(\frac{1}{3})$, $y(5)$.
6. If $y(x) = 3x - 2$, find $y(2x)$, $y(x - 1)$, $y(x^2)$, $y(t)$, $y(2t + 1)$.
7. Of $h(s) = s^2 - 1$, find $h(2x)$, $h(x^2)$, $h(t + 1)$, $h(\frac{1}{2}t)$, $h(t^2 + 1)$.
8. Given $F(t) = \dfrac{t + 1}{t - 1}$, find $F(0)$, $F(\frac{1}{2})$, $F(x^2)$, $F\left(\dfrac{1}{x}\right)$, $F\left(\dfrac{1}{x^2}\right)$.
9. Given $F(t) = \dfrac{t^2 + 1}{t^2 - 1}$, find $F(2)$, $F(-2)$, $F\left(\dfrac{1}{x}\right)$, $F\left(-\dfrac{3}{x}\right)$, $F\left(\dfrac{1}{x^2}\right)$.
10. Given $f(x) = 2x - 3$ and $y(x) = 3x + 4$, find $f(1) + y(2)$, $f(0) - y(3)$, $f(x) + y(x)$, $2f(3) + 5y(0)$, $f(4) \cdot y(1)$, $f(12) \div y(1)$.
11. Given $f(x) = 2x^2 + x - 3$ and $y(x) = 2x + 3$, find $f(x) \div y(x)$, $f(x) \cdot y(x)$.
12. Given $f(x) = x^2 - 4x - 5$, find $f(2 + h) - f(2)$, $f(h) - f(0)$.
13. If $f(x) = x^2 - 3$, find $\dfrac{f(3 + h) - f(3)}{h}$, $\dfrac{f(x + h) - f(x)}{h}$.
14. If $f(x) = \dfrac{1}{x}$, find $\dfrac{f(4 + h) - f(4)}{h}$, $\dfrac{f(x + h) - f(x)}{h}$.
15. Given $y(x) = x - 2$, find $2y(x)$, $y(x^2)$, $[y(x)]^2$, $3y(x)$, $y(x^3)$, $[y(x)]^3$.
16. Given $x - 2y = 5$, express (a) y as a function of x by solving for y; (b) x as a function of y by solving for x.
17. Given $xy - x + y + 1 = 0$, express each variable as a function of the other. This is done by solving the equation for each letter in terms of the other letter.
18. Express the area of a circle as a function of its radius.
19. Express the area of a triangle as a function of its base and altitude.
20. Express the volume of a cube as a function of its edge.
21. Express the area of the six faces of a cube as a function of its edge.

40. A Coordinate System. The relation between an independent variable and a function of the variable is a matter of great importance in mathematics. In the study of many problems it is helpful to represent the relation geometrically. The most common way of picturing the values of an independent variable and the corresponding values of a function of the variable is by means of the **rectangular coordinate system** described below.

Draw two perpendicular lines meeting at O (Fig. 2). The point O is called the **origin** and the lines the **axes**, OX the x-axis and OY the y-axis. The x-axis is usually referred to as the horizontal axis and the y-axis as the vertical axis. The axes divide their plane into four parts called **quadrants.** The quadrants are numbered I, II, III, and IV as shown in the figure. Next select any convenient unit of length and lay off distances from the origin along the axes. The distances to the right of O along the x-axis are defined as positive and those to the left of O are taken as negative. Similarly, distances upward of O along the y-axis are defined as positive and those downward of O are called negative.

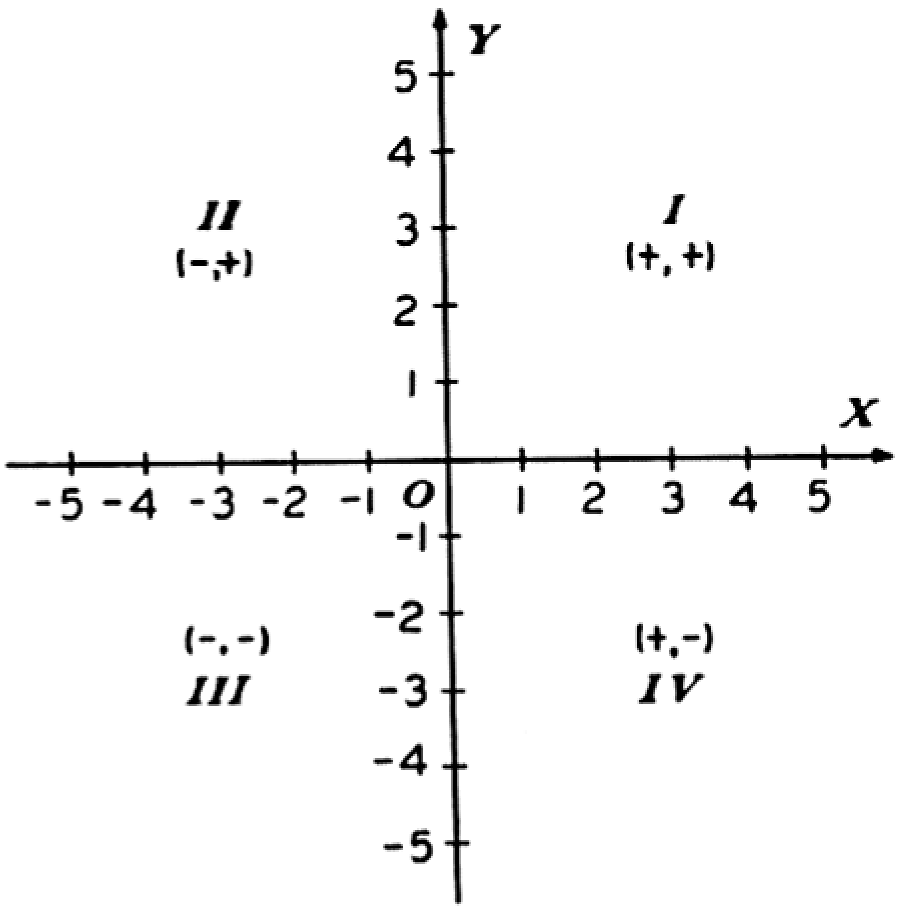


FIG. 2

The position of any point P in the plane may be definitely indicated by giving its distances from the axes. The distance from the y-axis to P is called the **abscissa** of the point, and the distance from the x-axis is called the **ordinate** of the point. The abscissa is positive if the point is to the right of the y-axis, and negative if the point is to the left of the y-axis. The ordinate is positive if the point is above the x-axis, and negative if the point is below the x-axis. The abscissa of a point on the y-axis is zero and the ordinate of a point on the x-axis is zero. The two distances, abscissa and ordinate, are called the **coordinates** of the point. The coordinates are indicated by writing the abscissa first and enclosing both numbers in parentheses. For example, $P(-2, 3)$, or just $(-2, 3)$, stands for the point whose abscissa is -2 and whose ordinate is 3. The coordinates of a point in the first quadrant are both positive; this is indicated in the figure by $(+, +)$. The signs of the coordinates in each of the other quadrants are similarly indicated.

To **plot** a point of given coordinates means to measure off the proper distances from the axes and to mark the point thus determined. To plot $(-2, 3)$, for example, go two units to the left from O along the x-axis and then go three units upward parallel to the y-axis.

In plotting points one may draw the axes on paper or the blackboard and estimate the distances by sight or measurement. But the work can be done more accurately and rapidly by using graph, or coordinate, paper.

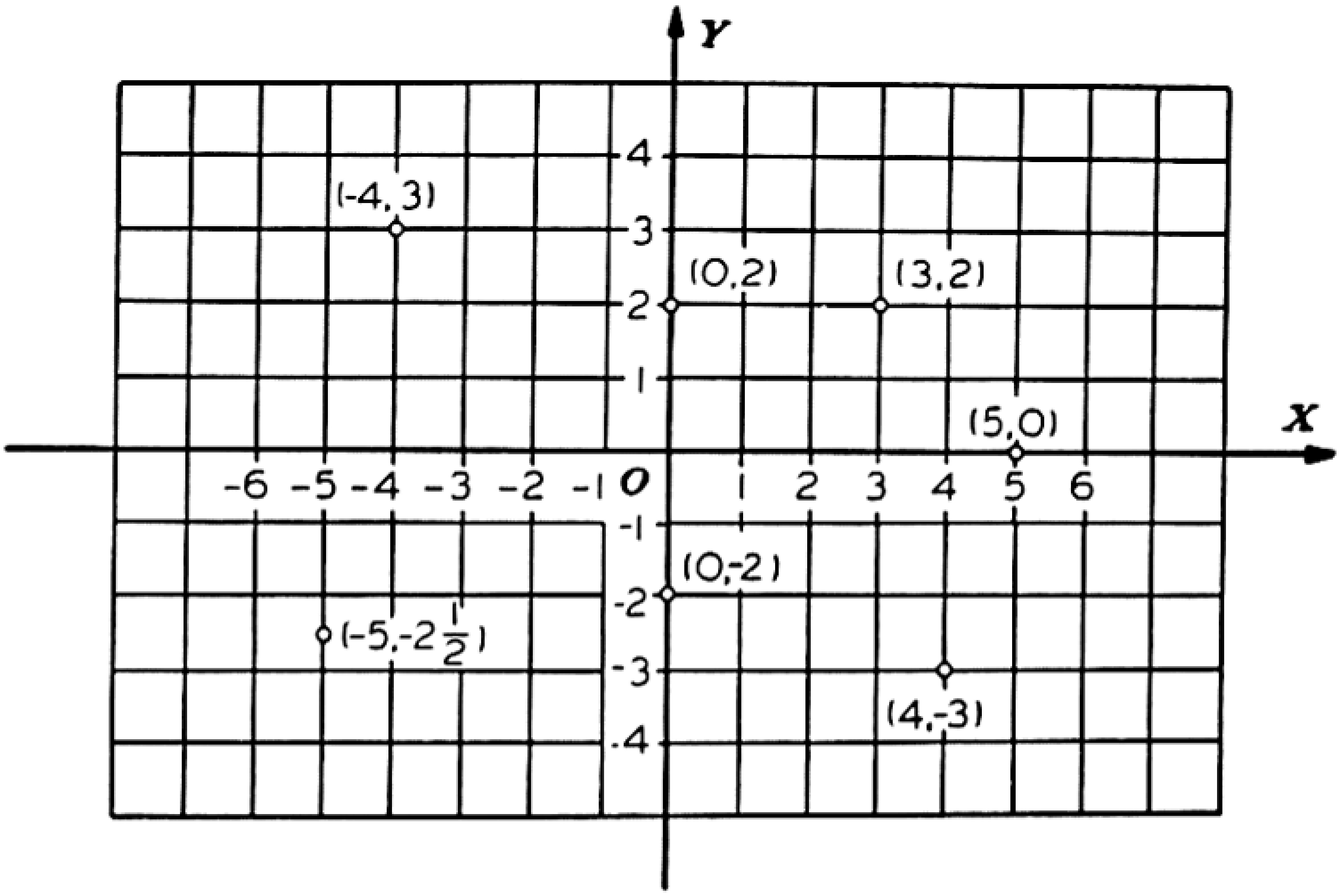


FIG. 3

This paper is ruled off into small squares. In Figure 3 the coordinates of some points are indicated. The coordinates should be verified by counting along the edges of the squares.

Exercise 23

Using graph paper, plot the following points:

1. $(3, 2)$.
2. $(5, -1)$.
3. $(-2, 2)$.
4. $(4, 0)$.
5. $(-3, 0)$.
6. $(-5, -2)$.
7. $(-3, -5)$.
8. $(5, 4)$.
9. $(\frac{7}{2}, 2)$.
10. $(3, \frac{2}{3})$.
11. $(\frac{3}{5}, \frac{7}{5})$.
12. $(-\frac{4}{5}, \frac{8}{5})$.
13. $(8, -\frac{11}{5})$.
14. $(7\frac{1}{2}, 0)$.
15. $(0, -5\frac{2}{7})$.
16. $(-\frac{9}{4}, \frac{5}{4})$.
17. $(0, 0)$.
18. $(-5, -5)$.

41. The Graph of a Function. We are now ready to illustrate the use of a coordinate system in representing a function graphically. For this purpose we choose the function $x^2 - 3x - 2$. Letting y stand for the function, we form the equation

$$y = x^2 - 3x - 2.$$

The first step is to assign x several values and compute the corresponding values of the function, or y. The value pairs are placed in columns so that each x-value has its corresponding y-value directly to the right. We take each value of x in the table and the corresponding value of y as the abscissa and ordinate of a point. The points thus determined are plotted in Figure 4. We now draw a smooth curve through the points going from point to point in the order of increasing x-values. The curve is called the **graph** of the given function, and the process of constructing the curve is called **graphing the function.**

x	y
$-\frac{3}{2}$	$\frac{19}{4}$
-1	2
$-\frac{1}{2}$	$-\frac{1}{4}$
0	-2
1	-4
$\frac{3}{2}$	$-\frac{17}{4}$
2	-4
$\frac{5}{2}$	$-\frac{13}{4}$
3	-2
4	2
$\frac{9}{2}$	$\frac{19}{4}$

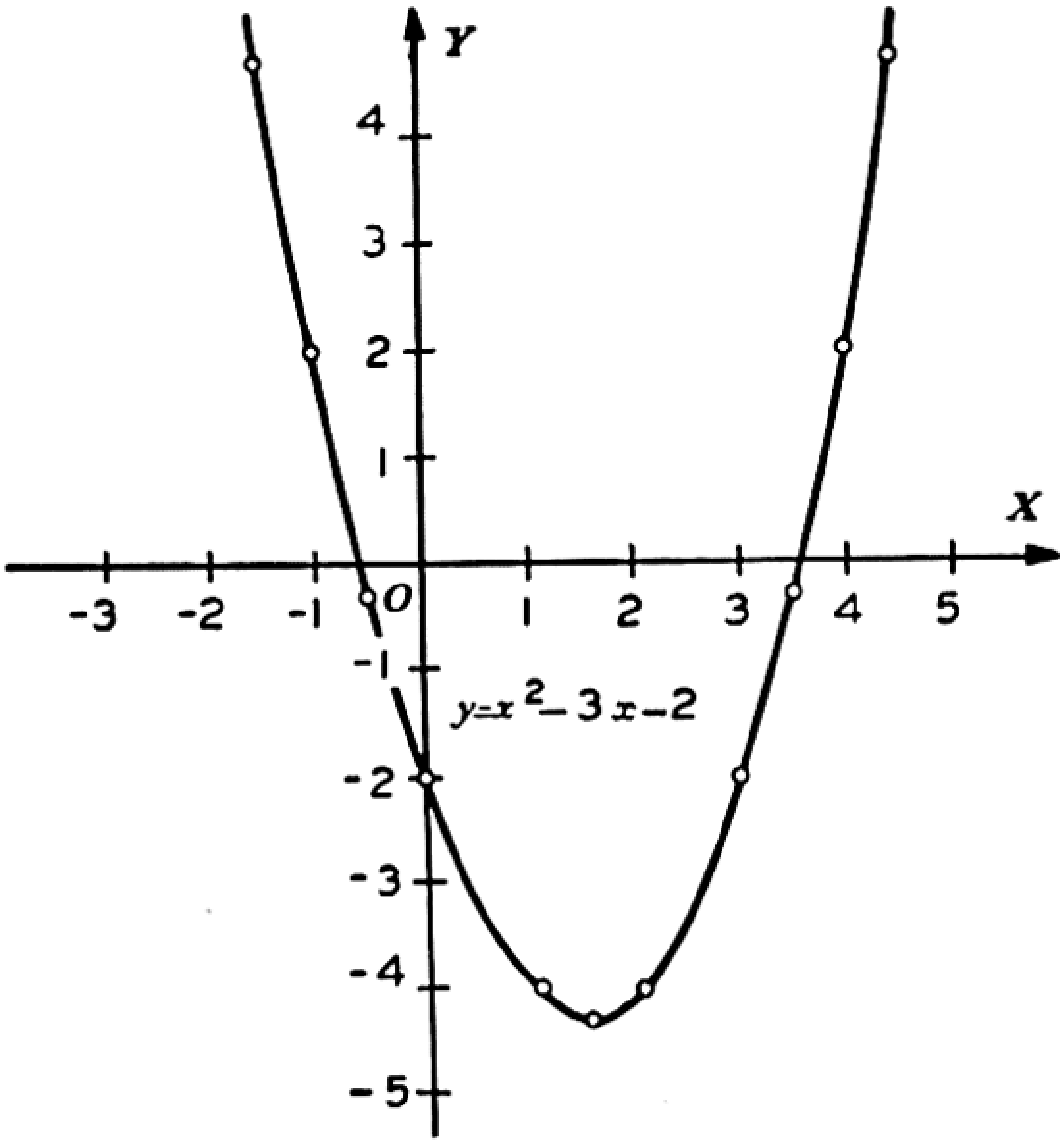


FIG. 4

In graphing the function we used x-values in the range -1.5 to 4.5. The range could be extended, but we are usually interested in studying functions in restricted intervals. The complete graph consists of all the points whose abscissas are values of the independent variable and whose ordinates are corresponding values of the function.

The point or points which the graph of a function has in common with the x-axis are of special interest. The abscissa of such a point is the value of x which makes y, or the function, equal to zero. A value of the independent variable which makes the function equal to zero is called a **zero of the function.** We estimate the zeros of the function pictured in Figure 4 to be -0.6 and 3.6.

Example. Draw the graph of the function $y = x^3 - 5x^2 + 15$ in the interval $x = -2$ to $x = 5$, and estimate the zeros of the function.

Solution. We substitute values for x and obtain the corresponding values of y. The values of the function y, as shown in the table, lie in the range -13 to 15. Taking note of these values, we select a scale along the y-axis so that all the points may be plotted on the area of paper to be used. The points determined by the tabulated values are plotted in Figure 5. These points serve as guides for drawing the graph.

x	y
-2	-13
-1	9
0	15
1	11
2	3
3	-3
4	-1
5	15

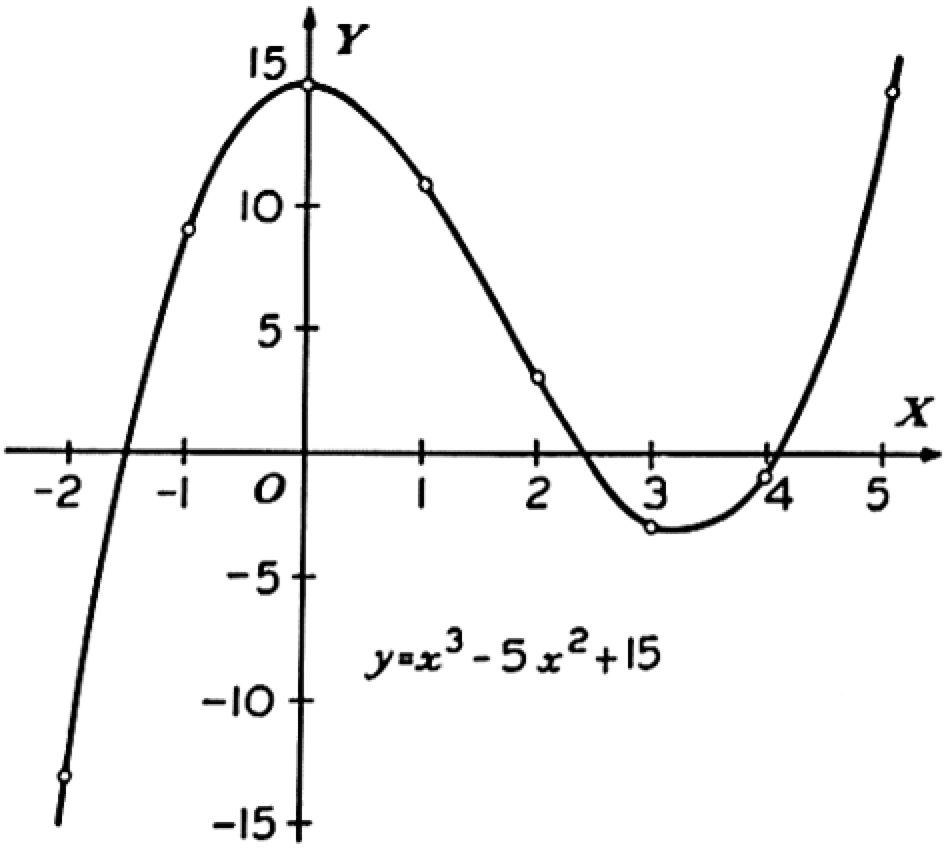


FIG. 5

The graph indicates that the given function has three zeros, which we estimate to be -1.5, 2.4, and 4.1. These values are approximately correct, as may be verified by substituting each for x in the function $x^3 - 5x^2 + 15$.

NOTE. In graphing functions it is necessary to notice from the table of values where to draw the coordinate axes on the graph paper, and what units of measurement to select along the axes in order to draw the graph on the page. It is usually better to have the curve cover a fairly large portion of the page. For some functions the same unit of measurement may well be used along both axes, but often the functional values have a wide range which necessitates a much smaller scale along its axis.

A decision also needs to be made as to what values to assign x. Inasmuch as the functions which we shall study in this chapter have graphs which are unlimited in length, we shall restrict the x-values in each case to an interval near the origin. The points determined by the integral values of x in the interval are sufficient for drawing a reasonably good graph of some functions. Sometimes, however, the points are too widely scattered vertically to suggest the shape of the graph. In such cases additional points should be found by assigning x a few fractional values.

42. The Graph of a Linear Function. An expression of the form $ax + b$, where b is any constant and a any constant except zero, is called a **linear** function of x. The graph of a linear function is a straight line. Proof of this fact is made in analytic geometry. It follows, therefore, that two points are sufficient for drawing the graph of a linear function. However, plotting a third point serves as a check against errors. It is also well, for greater accuracy in drawing the line, to have two of the points rather far apart.

Example. Draw the graph of $2x - 3$.

Solution. Denoting the function by y, we obtain the table of corresponding values. The graph of the function is a straight line, a segment of which is drawn in Figure 6.

x	y
0	-3
2	1
3	3

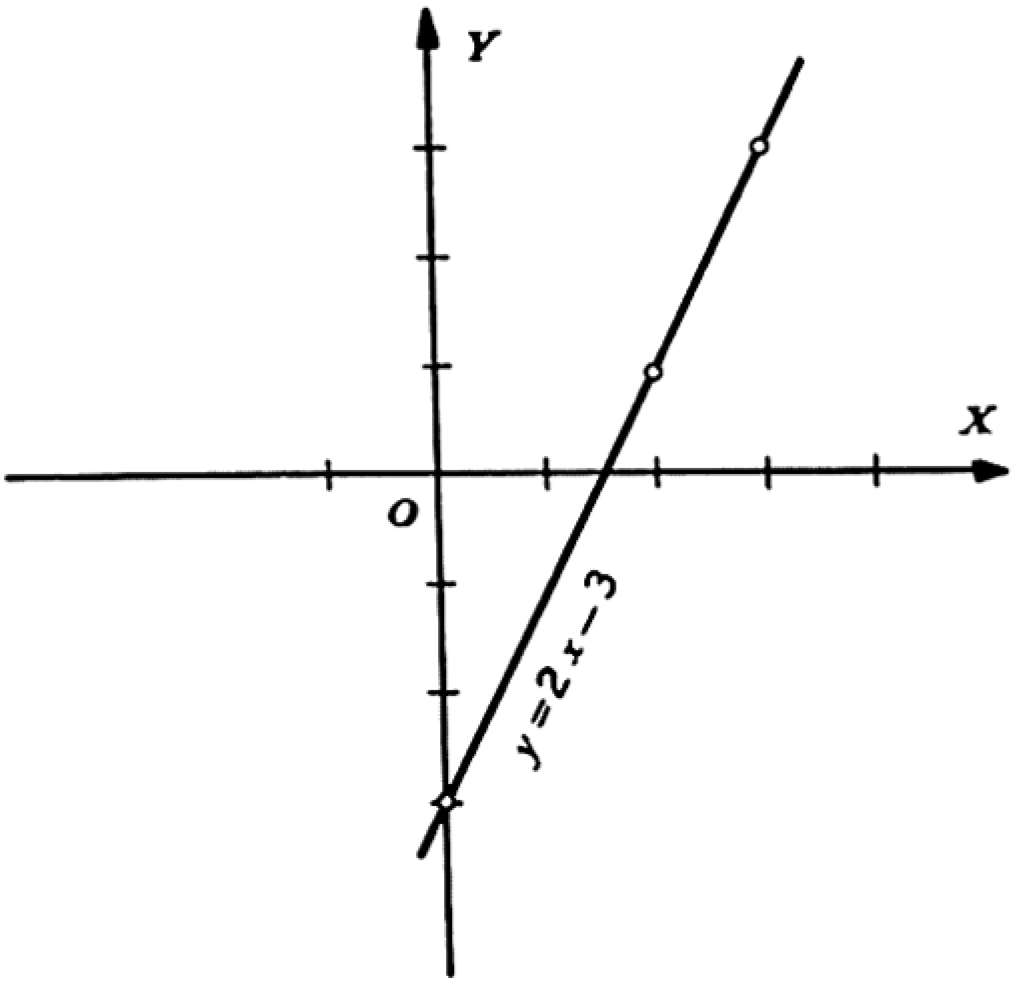


FIG. 6

Exercise 24

1. Plot carefully several points on the graph of the function $3x - 5$. Do the points appear to lie on a straight line?

2. Plot the points which have the coordinates $(-1, -5)$, $(0, 0)$, $(1, 3)$, $(1.5, 3.8)$, $(2, 4)$, $(2.5, 3.8)$, $(3, 3)$, $(4, 0)$, and $(5, -5)$. Draw a smooth curve through the points taken in the given order.

3. Determine if the points $(-1, 6)$, $(0, 0)$, $(1, -4)$, $(2, -6)$, $(2.5, -6.25)$, $(3, -6)$, $(4, -4)$, $(5, 0)$, and $(6, 6)$ are on the graph of the function $y = x^2 - 5x$. Plot the points and draw the graph.

Draw the graph of each of the following functions in the interval indicated. Plot the two end points and one intermediate point of each line segment.

4. $x - 3$, from $x = -3$ to $x = 8$. **5.** $2x + 5$, from $x = -5$ to $x = 1$.
6. $\frac{1}{2}x + 3$, from $x = -8$ to $x = 4$. **7.** $-x - 1$, from $x = -6$ to $x = 6$.
8. $-2x$, from $x = -4$ to $x = 4$. **9.** $-\frac{1}{4}x$, from $x = -8$ to $x = 8$.

Express y as a function of x by solving each equation **10–18** for y. Find three points on the graph and draw a line through the points.

10. $2x - y = 5$. **11.** $x - 2y = 4$. **12.** $3x + 2y + 6 = 0$.
13. $x + y = 0$. **14.** $4x - 2y = 1$. **15.** $x + 3y - 3 = 0$.
16. $2x - y = 0$. **17.** $x + 3y = 0$. **18.** $4x - 2y - 5 = 0$.

Draw the graph of each of the following functions in the interval indicated, and estimate its zeros. Use all the integral values of x in the interval and enough fractional values to determine the shape of the graph.

19. $y = x^2$, from $x - 3$ to $x = 3$.
20. $y = -x^2$, from $x = -3$ to $x = 3$.
21. $y = x^2 - 6x$, from $x = -2$ to $x = 8$.
22. $y = 2x^2 - 5$, from $x = -4$ to $x = 4$.
23. $y = -x^2 + 3x + 4$, from $x = -4$ to $x = 7$.
24. $y = 3x^2 - 6x - 4$, from $x = -2$ to $x = 4$.
25. $y = 2x^2 + 5x - 4$, from $x = -4$ to $x = 1.5$.
26. $y = \frac{1}{4}x^3$, from $x = -4$ to $x = 4$.
27. $y = -x^3$, from $x = -3$ to $x = 3$.

28. Hourly Fahrenheit temperature readings from 8:00 A.M. to 5:00 P.M. are as follows: 20°, 33°, 46°, 51°, 53°, 55°, 58°, 57°, 52°, 45°. Select convenient scales along the horizontal (time) axis and the vertical (temperature) axis and plot the points determined by these data. Make a graph by connecting the points consecutively with line segments.

29. The table gives the miles obtained per gallon of gasoline by a car at various speeds in miles per hour. Plot the points to represent these data. Draw a smooth curve through the points.

x (speed)	10	20	30	40	50	60	70
y (miles per gal.)	15	17.5	19	19.4	18.5	16	14

30. The table shows the distance which a car traveled while being brought to a stop from various speeds. The speeds are in miles per hour and the distances in feet. Construct a smooth curve to represent the stopping distance as a function of the speed.

x (speed)	10	20	30	40	50	60	70
y (distance)	3	12	27	48	75	108	147

Chapter Six

SYSTEMS OF LINEAR EQUATIONS

43. Linear Equations in Two Unknowns. An equation of the form

$$ax + by = c,$$

where a, b, and c are constants, with a and b not both zero, is called a **linear equation** in the unknowns x and y. A pair of numbers for x and y which satisfy the equation is said to be a **solution** of the equation. We may find a solution by substituting any number for one of the unknowns and solving for the other. For example, substituting $x = 2$ in the equation $3x - 4y = 18$, we find $y = -3$. Hence the pair of numbers $x = 2$, $y = -3$ is a solution of the equation. In this manner any number of solutions could be found.

Two linear equations in the same two unknowns taken together constitute a **system** of equations. A pair of values for x and y which satisfy both equations is called a solution of the system.

Example. Show that $x = 2$, $y = -3$ is a solution of the system

$$2x - 5y = 19$$

$$3x + 2y = 0.$$

Solution. Substituting in the two equations, we have

$$\begin{aligned} 2(2) - 5(-3) &= 19, & \qquad 3(2) + 2(-3) &= 0, \\ 4 + 15 &= 19, & 6 - 6 &= 0, \\ 19 &= 19. & 0 &= 0. \end{aligned}$$

The pair of numbers $x = 2$, $y = -3$ satisfy both equations and is a solution.

44. Solution by Graphing. There are different methods for solving linear equations in two unknowns. We shall first consider a graphical method.

If b is not zero, we can solve the equation $ax + by = c$ for y and get the equivalent equation

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Thus y is expressed as a linear function of x. The graph of the linear function is a straight line. Therefore, the coordinates of any point on the line constitute a solution of the equation obtained by solving for y and the equation as first written. We may conclude that the graph of two equations of the form $ax + by = c$ consists of two straight lines. If the lines intersect, the coordinates of the point of intersection satisfy each equation separately. Hence the solution of the system may be had by measuring off the coordinates of the intersection point.

Example 1. Solve graphically the system

$$\begin{aligned} 2x + 3y &= 6 \\ 3x - 10y &= 15. \end{aligned}$$

Solution. We prepare a table of values for each equation. This is done by assigning values to one of the unknowns and solving for the other unknown.

$2x + 3y = 6$

x	y
0	2
3	0
1.5	1

$3x - 10y = 15$

x	y
0	−1.5
5	0
2	−0.9

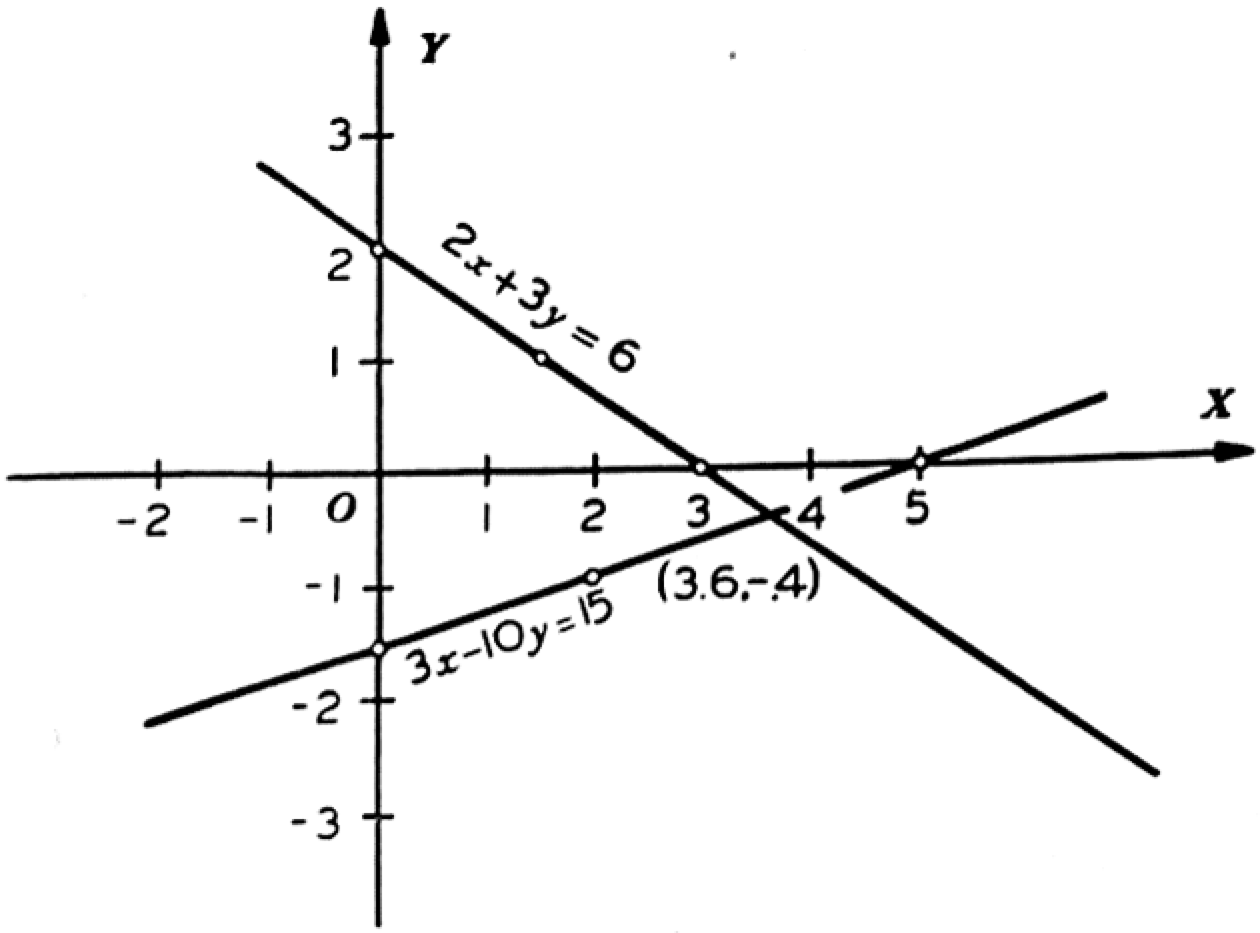


FIG. 7

The lines (Fig. 7) intersect. We estimate the coordinates of the intersection point as $x = 3.6$ and $y = -0.4$. These values are approximately correct. They satisfy the first equation but make the left side of the second equation 14.8 instead of 15.

Example 2. Graph the system

$$3x - 4y = 5$$
$$6x - 8y = -2.$$

Solution. We make a table of values for each equation.

$3x - 4y = 5$

x	y
-1	-2
3	1

$6x - 8y = -2$

x	y
-3	-2
1	1

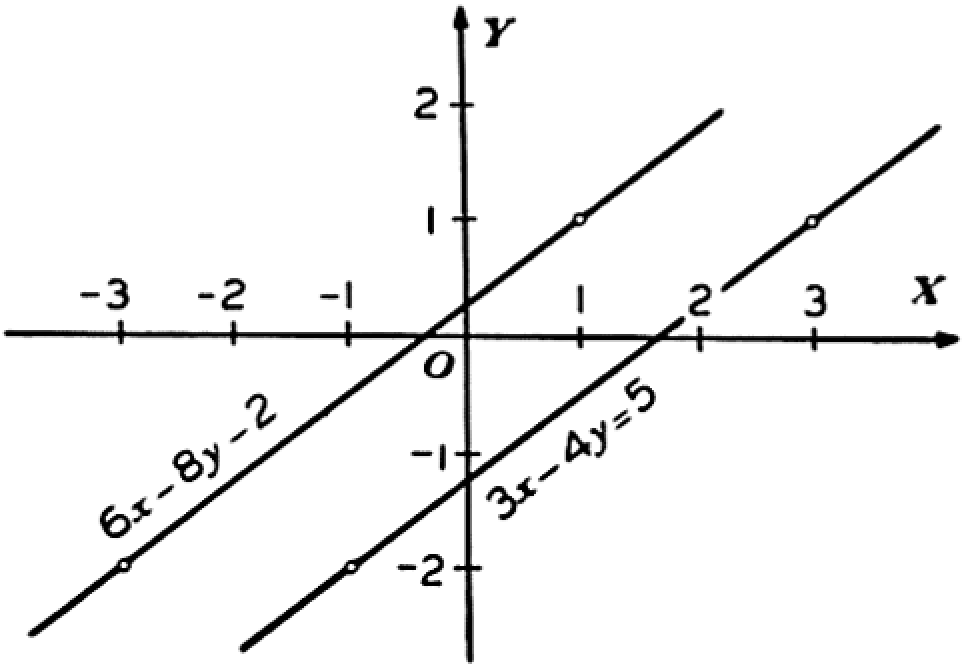


Fig. 8

The graph (Fig. 8) consists of two parallel lines. This means that there is no pair of numbers for x and y which satisfy the two equations simultaneously. It could be observed without graphing that the two equations have no solution. Multiplying both members of the first equation by 2, we may write the system as

$$6x - 8y = 10$$
$$6x - 8y = -2.$$

A pair of numbers which makes $6x - 8y$ equal to 10 would certainly not make $6x - 8y$ equal to -2.

Had the two equations been given as

$$3x - 4y = 5$$

$$6x - 8y = 10,$$

then the members of the first equation multiplied by 2 yield the second equation. Any solution of one equation would obviously be a solution of the other, and their graphs would coincide. For this case there is an unlimited number of solutions.

We have illustrated three different kinds of systems of linear equations. If the graphs of the equations are two intersecting lines, the system has one solution. The equations are said to be **consistent.** If the graphs are parallel lines, there is no solution. In this case the equations are **inconsistent.** The remaining case is that in which the graphs of the equations are coincident lines. There are then an unlimited number of solutions, and the equations are said to be **dependent.**

Inconsistent and dependent equations can be easily detected without drawing their graphs. To make the examination, write each equation with the unknowns in the left member and the constant term in the right member. If multiplying the members of one equation by a constant yields the second equation, the given equations are dependent. If the operation makes the left members alike and the right members different, the given equations are inconsistent.

Exercise 25

Solve graphically the systems which are consistent. For the others state if the equations are inconsistent or dependent, giving the reason for your answer.

1. $x + y = 5$
$x - y = 1.$

2. $x + 2y = 3$
$2x - y = -4.$

3. $5x + y = 6$
$3x + 2y = -2.$

4. $4x - 3y = 5$
$3x - 2y = 3.$

5. $6x - 7y = 27$
$2x + 3y = -7.$

6. $2x + 5y = 13$
$5x - 6y = 14.$

7. $x - 4y = 2$
$3x - 12y = 5.$

8. $7x + 3y = 4$
$14x + 6y = 8.$

9. $y - 4x = 3$
$2y - 8x = 6.$

10. $4y - 2x = 9$
$3y + 2x = 6.$

11. $7x - 6y = 13$
$3x - 5y = 10.$

12. $4x + 5y = 10$
$3x + 4y = 8.$

13. $4x + 6y = 5$
$6x + 9y = 7.$

14. $5x + 5y = 3$
$4x - 4y = 7.$

15. $5x - 3y = 7$
$2x + 5y = 3.$

16. $8x - 4y = 9$
$6x + 2y = 7.$

17. $3x - 3y = 5$
$2x - 2y = 5.$

18. $9x + 4y = -15$
$6x + 2y = -10.$

19. $x + 11y = 5$
$11x - 2y = 4.$

20. $5y - 4x = 3$
$6y - 3x = 2.$

21. $3x + 3y = 1$
$4x - 4y = 21.$

45. Solution by Algebraic Methods. We next discuss two different algebraic methods for solving a system of linear equations in two unknowns. These methods yield exact solutions; whereas, the graphical method, due to imperfect graphing and the imperfect estimation of coordinates, usually gives only approximate results.

A. *Elimination by Substitution.* We illustrate this method by solving the equations

$$3x + 2y = 3 \tag{1}$$

$$2x - 5y = -17. \tag{2}$$

Solution. Solving equation (1) for y, we get

$$y = \frac{3 - 3x}{2}.$$

By substituting this expression for y in equation (2), we obtain a linear equation which can be solved for x. That is,

$$2x - 5\left(\frac{3 - 3x}{2}\right) = -17.$$

Multiplying by 2, we have

$$4x - 15 + 15x = -34,$$

whence

$$19x = -19,$$

$$x = -1.$$

Then

$$y = \frac{3 - 3x}{2} = \frac{3 - 3(-1)}{2} = \frac{3 + 3}{2} = 3.$$

Hence, the solution is $x = -1$, $y = 3$. We check by substituting these values in the left members of the given equations.

Check:

$$(1) \quad 3(-1) + 2(3) = -3 + 6 = 3.$$

$$(2) \quad 2(-1) - 5(3) = -2 - 15 = -17.$$

The substitution method involves the following:

1. Solve either of the equations for one of the unknowns in terms of the other.

2. Substitute the result in the other equation. Solve the equation thus obtained for its unknown.

3. Substitute in either of the given equations to find the remaining unknown.

(The values for the unknowns should be checked in the original equations.)

B. *Elimination by Addition or Subtraction.* We apply this method in solving the system

$$5x + 3y = -26 \tag{1}$$

$$4x - 9y = 2. \tag{2}$$

Solution. If we multiply the members of equation (1) by 4 and those of equation (2) by 5, the coefficients of x in the resulting equations are the same. Thus we get

$$20x + 12y = -104 \tag{3}$$

$$20x - 45y = 10. \tag{4}$$

We subtract corresponding members of equations (3) and (4) to eliminate x. This gives

$$57y = -114,$$

$$y = -2.$$

Substituting -2 for y in equation (1) yields

$$5x - 6 = -26,$$

$$5x = -20,$$

$$x = -4.$$

The solution is $x = -4$, $y = -2$. We check by substituting these values in the left members of the given equations.

Check:

$$5(-4) + 3(-2) = -20 - 6 = -26$$

$$4(-4) - 9(-2) = -16 + 18 = 2.$$

The following steps are involved in the addition or subtraction method:

1. Multiply the members of one or both equations by properly chosen constants and obtain equivalent equations which have the coefficients of one of the unknowns numerically equal.

2. Then by adding or subtracting eliminate that unknown.

3. Solve the resulting equation for its unknown.

4. Substitute the number thus found in either of the original equations and find the other unknown.

Step 1 should be omitted if the given equations have the coefficients of either unknown numerically equal.

When both unknowns are eliminated simultaneously, the equations are either dependent or inconsistent. If the elimination process leads to a numerical equality, the equations are dependent; if to a numerical inequality, the equations are inconsistent.

Exercise 26

Solve each pair of equations, if found consistent, by the method of substitution. Then check your solution by using the addition or subtraction method.

1. $2x - 3y = 2$
$4x + 3y = 22.$

2. $3x + y = 8$
$5x - y = 16.$

3. $x + y = 1$
$3x - y = 3.$

4. $4x + 3y = 1$
$2x + 5y = 11.$

5. $x - 6y = 5$
$6x - y = -5.$

6. $3x - 4y = 3$
$6x - 8y = 6.$

7. $2x - 7y = 17$
$4x - 5y = 25.$

8. $5x + 3y = 8$
$4x + 2y = 6.$

9. $7x - 2y = 16$
$5x + 8y = 2.$

10. $15x + 3y = 2$
$10x + 2y = 3.$

11. $6x - 4y = 0$
$12x - 8y = 0.$

12. $3y - 5x = 6$
$4y + 6x = -30.$

13. $3x - 4y = 7$
$5x - 5y = 8.$

14. $7x + 3y = 4$
$4x + 2y = 3.$

15. $x + 2y = 2$
$12x - 15y = -11.$

16. $3x - 4y = 5$
$6x - 7y = 10.$

17. $2x - 4y = -3$
$4x + 2y = 9.$

18. $12x - 12y = 13$
$4x + 6y = 1.$

19. $5y - 4x = 3$
$4x - 5y = 2.$

20. $5x + 7y = -2$
$4x + 6y = -3.$

21. $8x - 9y = 10$
$4x + 3y = 4.$

46. Fractional Equations. Frequently we have fractional equations which yield linear equations when cleared of fractions. The solution of a pair of equations of this kind in two unknowns can be found by first clearing of fractions and then applying one of the two methods of Section 45.

Example 1. Solve the system

$$\frac{2x + y}{3x - y + 1} = \frac{5}{16}$$

$$\frac{x - 3}{y + 5} = \frac{1}{2}.$$

Solution. Multiplying the members of the first equation by $16(3x - y + 1)$, we get

$$32x + 16y = 15x - 5y + 5,$$

$$17x + 21y = 5.$$

Clearing the second equation of fractions yields

$$2x - 6 = y + 5,$$

$$2x - y = 11.$$

The equations $17x + 21y = 5$ and $2x - y = 11$ have the solution $x = 4$, $y = -3$. We check these values by substituting in the left members of the given equations.

Check:

$$\frac{2(4) + (-3)}{3(4) - (-3) + 1} = \frac{8 - 3}{12 + 3 + 1} = \frac{5}{16}$$

$$\frac{4 - 3}{-3 + 5} = \frac{1}{2}.$$

For the case in which the fractional equations are linear in the reciprocals of the unknowns, solve first for the reciprocals without clearing of fractions.

Example 2. Solve the system

$$\frac{2}{x} - \frac{3}{y} = -1 \tag{1}$$

$$\frac{3}{x} - \frac{5}{y} = \frac{-11}{6}. \tag{2}$$

Solution. Multiply the members of equation (1) by 3 and those of equation (2) by 2. This gives

$$\frac{6}{x} - \frac{9}{y} = -3 \tag{3}$$

$$\frac{6}{x} - \frac{10}{y} = \frac{-11}{3}. \tag{4}$$

Subtracting equation (4) from equation (3), member from member, we get

$$\frac{1}{y} = -3 + \frac{11}{3} = \frac{2}{3}.$$

We now substitute 2 for $3/y$ in equation (1) and obtain

$$\frac{2}{x} - 2 = -1,$$

$$\frac{1}{x} = \frac{1}{2}.$$

Since the reciprocals of the unknowns are $\frac{1}{2}$ and $\frac{2}{3}$, the solution is $x = 2$, $y = \frac{3}{2}$.

47. Equations with Literal Coefficients. The methods used in Section 45 are also applicable to linear equations which have literal coefficients.

Example. Considering a and b as constants and x and y as the unknowns, solve the system

$$ax + by = 2a \tag{1}$$

$$3ax - 4by = -a + 7b. \tag{2}$$

Solution. To eliminate y multiply the members of equation (1) by 4 and leave equation (2) unchanged.

$$4ax + 4by = 8a$$

$$3ax - 4by = -a + 7b.$$

Adding, we have

$$7ax = 7a + 7b,$$

$$x = \frac{a+b}{a}.$$

Now substitute $a + b$ for ax in equation (1). This gives

$$a + b + by = 2a,$$

$$y = \frac{a-b}{b}.$$

The solution is $x = \dfrac{a+b}{a}$, $y = \dfrac{a-b}{b}$.

Check:

$$a\left(\frac{a+b}{a}\right) + b\left(\frac{a-b}{b}\right) = a + b + a - b = 2a$$

$$3a\left(\frac{a+b}{a}\right) - 4b\left(\frac{a-b}{b}\right) = 3a + 3b - 4a + 4b = -a + 7b.$$

Exercise 27

Solve for the unknowns occurring in each pair of equations:

1. $\dfrac{1}{2x - 3y + 3} = \dfrac{1}{14}$
$x - y = 4.$

2. $\dfrac{x-2}{y+2} = \dfrac{x+1}{y}$
$x - 3y = -2.$

3. $\dfrac{2x - 2y + 5}{x - y + 4} = 3$
$x + 2y = 2.$

4. $\dfrac{2x + y - 4}{3x - y - 1} = \dfrac{3}{2}$
$3x - 4y = 0.$

5. $\dfrac{3x - y - 9}{2x + 3y + 1} = 2$
$2x - 3y = 12.$

6. $\dfrac{2}{x + 3y + 3} = \dfrac{1}{2}$
$2x + 3y = 5.$

7. $\dfrac{x-y+3}{2x+y-3}=\dfrac{3}{2}$
$\dfrac{3x-2y+4}{2x-2y+3}=\dfrac{5}{2}.$

8. $\dfrac{x-y-1}{x+y+2}=-\dfrac{4}{3}$
$\dfrac{x+2y-1}{2x-3y+2}=-\dfrac{1}{3}.$

9. $\dfrac{x-2y+3}{2x-3y}=\dfrac{4}{3}$
$\dfrac{2x+5y-8}{x-y}=\dfrac{3}{2}.$

10. $\dfrac{2x+2}{2y-3}=\dfrac{x-2}{y+3}$
$\dfrac{x+2}{3y-2}=\dfrac{x-1}{3y+8}.$

11. $\dfrac{2x-3}{2x-1}=\dfrac{y+7}{y+5}$
$\dfrac{5x-4}{x+5}=\dfrac{5y-14}{y+3}.$

12. $\dfrac{x+8}{y+7}=\dfrac{x-6}{y-5}$
$\dfrac{2y-3}{2x-5}=\dfrac{y+8}{x+8}.$

13. $\dfrac{1}{x}-\dfrac{2}{y}=3$
$\dfrac{1}{x}+\dfrac{3}{y}=2.$

14. $\dfrac{2}{x}+\dfrac{3}{y}=1$
$\dfrac{4}{x}+\dfrac{3}{y}=4.$

15. $\dfrac{2}{x}+\dfrac{5}{y}=9$
$\dfrac{4}{x}+\dfrac{4}{y}=-\dfrac{3}{2}.$

16. $\dfrac{3}{x}+\dfrac{9}{y}=-1$
$\dfrac{2}{x}-\dfrac{5}{y}=3.$

17. $\dfrac{3}{x}-\dfrac{5}{y}=7$
$\dfrac{4}{x}-\dfrac{3}{y}=2.$

18. $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{2}{3}$
$\dfrac{3}{x}-\dfrac{5}{y}=\dfrac{2}{3}.$

19. $\dfrac{3}{5x}+\dfrac{1}{2y}=\dfrac{1}{10}$
$\dfrac{5}{3x}+\dfrac{1}{y}=\dfrac{2}{3}.$

20. $\dfrac{1}{2x}-\dfrac{2}{3y}=\dfrac{2}{3}$
$\dfrac{3}{2x}-\dfrac{1}{3y}=\dfrac{3}{2}.$

21. $\dfrac{1}{3x}+\dfrac{3}{4y}=\dfrac{5}{2}$
$\dfrac{5}{9x}+\dfrac{2}{3y}=3.$

Solve each system of equations for x and y:

22. $3x+ay=4$
$2x+ay=2.$

23. $4x-3y=3a+4b$
$ax-2by=3ab.$

24. $x-y=2a$
$x+y=2b.$

25. $ax+y=1$
$bx-y=2.$

26. $ax-by=2b$
$3ax+4by=7a-b.$

27. $ax+by=c$
$x+y=1.$

28. $ax-by=1$
$cx+dy=1.$

29. $bx-2ay=3b-a$
$3bx+3ay=6a.$

30. $3ax-2by=b$
$2ax+3by=a.$

31. $ax+by=c$
$dx+ey=f.$

32. $ax-by=a^2-b^2$
$bx-ay=a^2-b^2.$

33. $ax-3by=c$
$2ax+2by=9c.$

48. Linear Equations in Three Unknowns. We next consider the problem of solving three linear equations in three unknowns. The algebraic methods which were used for solving two equations in two unknowns will serve in an analogous fashion to yield the solution for the case of three unknowns. The following steps may be employed:

1. Choose a pair of equations from the three given equations and eliminate one of the unknowns.

2. Select one of the pair and the unused equation and eliminate the same unknown.

3. Solve the two equations resulting from steps 1 and 2 for the two unknowns involved.

4. Substitute these values in one of the given equations to obtain the remaining unknown.

A system of three linear equations in three unknowns will have one solution, no solution, or infinitely many solutions. This same situation was discussed in the case of two unknowns. But we shall consider systems of three unknowns which have one and only one solution.

Example. Solve the system

$$2x + 3y + z = 9 \tag{1}$$

$$x - 2y - 3z = 1 \tag{2}$$

$$5x + 4y + 6z = 5. \tag{3}$$

Solution. We select x as the unknown to be eliminated. First rewrite equation (1) and multiply both sides of equation (2) by 2.

$$2x + 3y + z = 9$$

$$2x - 4y - 6z = 2.$$

Subtracting, we have

$$7y + 7z = 7,$$

$$y + z = 1.$$

Next rewrite equation (3) and multiply the members of equation (2) by 5.

$$5x + 4y + 6z = 5$$

$$5x - 10y - 15z = 5.$$

Subtracting gives

$$14y + 21z = 0,$$

$$2y + 3z = 0.$$

We have now obtained the equations $y + z = 1$ and $2y + 3z = 0$. The solution of these equations is $y = 3$, $z = -2$. Substituting these values for y and z in equation (2), we have

$$x - 2(3) - 3(-2) = 1 \quad \text{and} \quad x = 1.$$

Hence the solution of the given system is $x = 1$, $y = 3$, $z = -2$. These values satisfy each of the given equations.

Exercise 28

Solve for x, y, and z or u, v, and w. Check each problem by substituting your results in the given equations.

1. $2x - y + 3z = 9$
$3x + y + 2z = 11$
$x - y + z = 2.$

2. $x - 4y + 5z = -4$
$x + 3y + z = 6$
$2x - 3y + 2z = -6.$

3. $3x - 4y - z = 1$
$x - y + 3z = 3$
$3x - 2y + 2z = 0.$

4. $x - y + 2z = 8$
$3x + y - 3z = 2$
$2y + 9z = 16.$

5. $5x + 4y + 3z = 1$
$2x - y - z = 7$
$3x + y = 6.$

6. $4x - 3y + 2z = 2$
$3x - 2y + z = 3$
$2x + y = 4.$

7. $2u - 3v + 4w = 6$
$3u + 2v - 3w = 9$
$u + 2v + 4w = 3.$

8. $6u + 2v - w = -1$
$2u + 3v - 3w = 0$
$u - 2v - 2w = 4.$

9. $4u + v + 2w = 10$
$3u + 2v + w = 5$
$2u + 3v + 2w = 10.$

10. $u - v = 1$
$u + w = -1$
$v - w = 6.$

11. $2u + 3v = 2$
$u + 2w = 1$
$3v + 8w = 3.$

12. $4u - 9v = 4$
$u - 6w = 4$
$3v - 4w = 1.$

13. $2x - y + z = 0$
$x + y - 2z = 0$
$x + y + z = a.$

14. $ax + by = 2$
$ax + 2cz = 3$
$by - 3cz = -2.$

15. $x + y + z = 2a$
$x - y + z = 2b$
$x + y - z = 2c.$

16. $\frac{1}{u} - \frac{1}{w} = \frac{3}{2}$
$\frac{1}{v} - \frac{1}{w} = \frac{4}{3}$
$\frac{1}{u} + \frac{1}{v} = \frac{5}{6}.$

17. $\frac{1}{u} - \frac{1}{v} = -5$
$\frac{2}{u} - \frac{1}{w} = -2$
$\frac{3}{v} + \frac{1}{w} = 2.$

18. $\frac{2}{u} - \frac{3}{v} = 2$
$\frac{3}{u} - \frac{5}{w} = -1$
$\frac{3}{v} + \frac{2}{w} = 0.$

19. $\frac{2}{x} + \frac{2}{y} + \frac{3}{z} = 4$
$\frac{1}{x} + \frac{4}{y} - \frac{4}{z} = 0$
$\frac{3}{x} + \frac{3}{y} + \frac{2}{z} = 1.$

20. $\frac{3}{x} + \frac{4}{y} + \frac{2}{z} = 8$
$\frac{6}{x} + \frac{2}{y} + \frac{1}{z} = 7$
$\frac{9}{x} - \frac{2}{y} - \frac{3}{z} = 0.$

21. $\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = \frac{4}{3}$
$\frac{2}{x} - \frac{3}{y} + \frac{2}{z} = \frac{5}{3}$
$\frac{3}{x} - \frac{4}{y} + \frac{6}{z} = 3.$

22. $ax + by + cz = 6$
$ax - by + cz = 8$
$ax - by - cz = 0.$

23. $3x + y + 3z = 9$
$4x + y + 2z = 13$
$7x + 2y - z = 24.$

24. $2x - 3y - 2z = 3$
$3x - 2y + 3z = -3$
$4x - 4y + z = -1.$

25. $3u + 3v - 7w = 5$
$2u + 2v - 2w = 3$
$4u - 3v - 6w = 1.$

26. $2u + 3v + 3w = 1$
$3u - 2v - 2w = 1$
$2u - 4v - 5w = 1.$

27. $5u + 2v - 4w = 8$
$3u - 3v - w = 5$
$4u - 5v + 7w = 1.$

28. $4x + 2y + 3z = 3$
$3x - 6y + 4z = 3$
$x + 2y + z = 1.$

29. $5x + 4y + 3z = 3$
$4x - 3y - 2z = -1$
$6x - 4y - 2z = -1.$

30. $x + 2y + 4z = -1$
$3x - y + 3z = -1$
$5x - 4y - 3z = 0.$

49. Stated Problems Leading to Systems of Linear Equations. There are many kinds of worded problems involving two or more unknowns which can be solved by setting up a system of linear equations from the known data. The reasoning involved in obtaining the equations is much like that needed for the equations of one unknown in Section 36. We shall illustrate by solving some examples.

Example 1. A rectangle whose length exceeds its width by 2 feet has the same area as a rectangle which is 3 feet narrower and 4 feet longer. Find the dimensions of the first rectangle.

Solution.

Let x = the length, and
y = the width of the first rectangle. Then
$x + 4$ = the length, and
$y - 3$ = the width of the second rectangle.

We obtain one equation by noting that the length of the first rectangle exceeds its width by 2 feet; hence,

$$x - y = 2.$$

The second equation is obtained by equating the two areas; that is,

$$xy = (x + 4)(y - 3),$$

or

$$xy = xy - 3x + 4y - 12,$$

and

$$3x - 4y = -12.$$

Then, solving the system

$$\begin{aligned} x - y &= 2 \\ 3x - 4y &= -12, \end{aligned}$$

we obtain $x = 20$ feet and $y = 18$ feet, the dimensions of the first rectangle. (See Section 36, Example 1.)

Example 2. A and B can do a piece of work in 9 days. A and C can do the work in 8 days, and B and C can do the work in 12 days. Find how long it would take each working alone to do the work.

Solution.

Let x = the number of days required for A do to the work alone, and
y = the number of days for B alone, and
z = the number of days for C alone.

Then $1/x$, $1/y$, and $1/z$ give the fractional parts of the work done in one day, respectively, by A, B, and C. Therefore, we may write the equations

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{9}$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{8}$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{12}.$$

These equations yield

$$x = 13\tfrac{1}{11}, \quad y = 28\tfrac{4}{5}, \quad z = 20\tfrac{4}{7}.$$

Exercise 29

Use two of three unknowns in solving the problems of this exercise.

1. The sum of two numbers is 98 and their difference is 30. Find the numbers.

2. The difference of two numbers is 13, and twice the smaller is 1 more than the larger. Find the numbers.

3. The length of a rectangle exceeds its width by 4 inches. The perimeter is 50 inches. Find the dimensions of the rectangle.

4. A rectangle whose length exceeds its width by 9 feet has 44 square feet less area than a rectangle 3 feet longer and 2 feet wider. Find the dimensions of the rectangles.

5. If the numerator of a certain fraction is decreased by 1 and the denominator increased by 2, the value of the new fraction is $\frac{2}{5}$. But the value of the new fraction would be $\frac{2}{3}$ if both the numerator and denominator are increased by 1. Find the original fraction.

6. The sum of two numbers is 60, and the larger divided by the smaller produces the quotient 3 and remainder 8. Find the numbers.

7. The rate of a boat upstream on a river is 6 miles per hour, and the rate downstream is 11 miles per hour. Find the speed of the boat in still water and the rate of the river's current.

8. An airplane travels 360 miles with the wind in $1\frac{1}{2}$ hours and returns in 2 hours against the wind. Find the speed of the airplane in still air and the speed of the wind.

9. Two men, starting at a point on a 1-mile circular track, go in opposite directions with uniform speeds and meet in 6 minutes. But if they go in opposite directions, it requires 2 hours for the faster to gain a lap. Find their speeds in miles per hour.

10. A and B working together can do a piece of work in 9 days. A and C can do the work in 8 days. And B and C can do the work in 12 days. Find how long it would take each man working alone to do the work.

11. A certain job can be done by A and B in 12 hours, by A and C in 10 hours, and by B and C in 20 hours. Find how long it would take each to do the job alone.

12. A, B, and C working together can do a task in 8 hours. A and B can do the work in 12 hours, and B and C can do the task in 11 hours. How long would it take each man to do the work alone?

13. Two grades of coffee, one worth 85 cents per pound and the other 92 cents per pound, are to be mixed to make 100 pounds worth 90 cents per pound. How many pounds of each grade should be used?

14. How much milk, 3% butterfat, and how much cream, 30% butterfat, should be mixed to make 500 gallons of milk which is 4% butterfat?

15. A baseball club has won 55% of its games. By winning 10 of the next 15 games the rating will be raised to 60%. Find how many games the club has played and how many of these were won.

16. A father is three times as old as his son. In 14 years he will be twice as old as his son. Find their present ages.

17. A father is twice as old as his son. In 10 years the ratio of their ages will be 3 to 5. Find their present ages.

18. A man loans $8,000, part at 3% annual interest and the rest at $3\frac{1}{2}$%. The total annual income from the investments is $268. Find the amount of each investment.

19. A man makes three investments totaling $12,000. The interest rates are $2\frac{1}{2}$%, 3%, and $3\frac{1}{2}$% per year. The total annual income is $370, and the income from the 3% investment is $10 less than the combined income from the other investments. Find the amount of each investment.

20. A man weighing 160 pounds sits on a teeter board 10 feet from the fulcrum (point of support). The board is kept in balance by two boys who sit on the other end at distances 8 and 10 feet from the fulcrum. When the boys exchange places the balance is preserved if the man takes a 4-pound weight. Find the weight of each boy.

Hint: Multiply each weight on one end by its distance from the fulcrum. Add these products and set the sum equal to the corresponding sum for the other end.

21. Two children of weights 60 and 80 pounds sit on the ends of a 14-foot teeter board. Where should the fulcrum be placed for the board to be in balance?

22. A teeter board is in balance with weights of 120 and 200 pounds at the ends. If a 64-pound weight is placed 4 feet from the 120-pound weight, the fulcrum must be moved 1 foot to preserve the balance. Find the length of the board.

23. A baseball club has won 45% of its games. If the team can capture 32 of the next 54 games, the rating will then be 50%. Find the number of games already played and the number won.

24. A merchant has $30 worth of dimes and quarters. How many coins are there of each kind if the total number is 183?

25. The sum of three numbers is 33. The sum of the first two is 1 less than the third, and the sum of the first and third is 11 more than the second. Find the three numbers.

50. Determinants. Systems of linear equations are sometimes solved by using determinants. We shall define and discuss the determinants which are needed for solving linear systems of two and three unknowns.

A combination of four numbers arranged in a square with vertical lines on either side is called a **determinant of second order.** The four numbers form two rows and two columns. The determinant stands for the value obtained by multiplying the number in the first row and first column by the diagonally opposite number and then subtracting the product of the numbers on the other diagonal.

Illustrations:

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 3\cdot 5 - 4\cdot 2 = 15 - 8 = 7.$$

$$\begin{vmatrix} 3 & 4 \\ -2 & 6 \end{vmatrix} = 3\cdot 6 - (-2)(4) = 18 + 8 = 26.$$

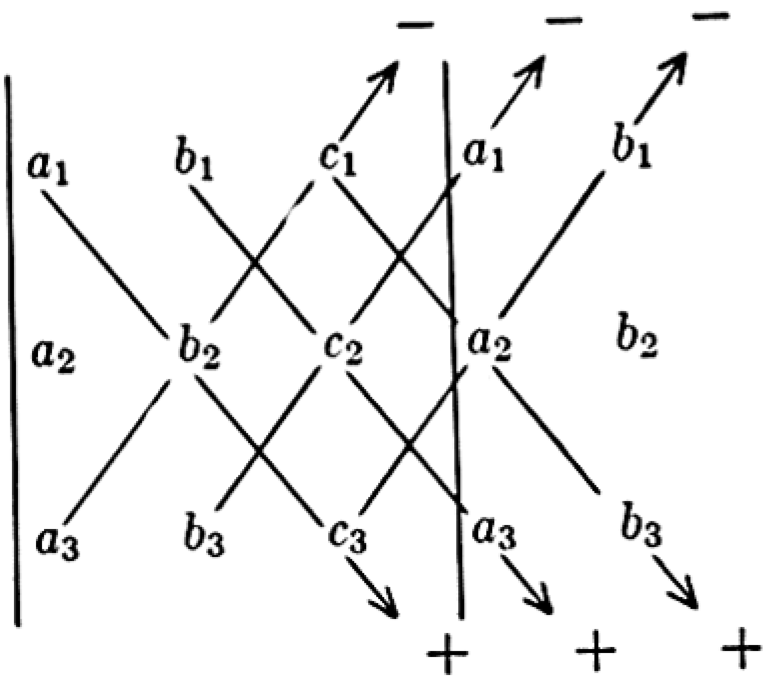
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Nine numbers arranged in a square of three rows and three columns constitute the numbers of a **determinant of third order.** The determinant stands for six terms called its **expansion.** We show here a determinant and its expansion.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

Each term of the expansion has three factors. And it may be observed that the three factors consist of one and only one number from each row and each column of the determinant. Further, a certain three of the terms are given plus signs and the others are given minus signs. The expansion of any third order determinant may be written by using this determinant and its expansion as a guide. To obviate the need for burdensome memorization, however, we describe a simple device for writing the expansion. The device, which applies only to determinants of the third order, has the following steps.

1. To the right of the determinant repeat the first and second columns.



2. The products of the three numbers in each of the three diagonals running to the right and downward give three terms of the expansion.

3. The negatives of the products of the three numbers in each of the three diagonals running to the right and upward are the other three terms of the expansion.

Illustrations:

$$\begin{vmatrix} 2 & 5 & 2 \\ 1 & -3 & 1 \\ -4 & 1 & -5 \end{vmatrix} = \left|\begin{array}{rrr|rr} 2 & 5 & 2 & 2 & 5 \\ 1 & -3 & 1 & 1 & -3 \\ -4 & 1 & -5 & -4 & 1 \end{array}\right.$$

$$\begin{aligned} &= (2)(-3)(-5) + (5)(1)(-4) + (2)(1)(1) \\ &\qquad -(-4)(-3)(2) - (1)(1)(2) - (-5)(1)(5) \\ &= 30 - 20 + 2 - 24 - 2 + 25 = 11. \end{aligned}$$

$$\begin{vmatrix} 12 & -1 & 5 \\ 3 & 2 & -7 \\ 1 & -3 & 1 \end{vmatrix} = \left|\begin{array}{rrr|rr} 12 & -1 & 5 & 12 & -1 \\ 3 & 2 & -7 & 3 & 2 \\ 1 & -3 & 1 & 1 & -3 \end{array}\right.$$

$$= 24 + 7 - 45 - 10 - 252 + 3 = -273.$$

Exercise 30

Evaluate the determinants:

1. $\begin{vmatrix} 4 & 3 \\ 7 & 9 \end{vmatrix}.$ **2.** $\begin{vmatrix} 5 & -7 \\ 4 & 3 \end{vmatrix}.$ **3.** $\begin{vmatrix} 9 & 8 \\ 3 & -5 \end{vmatrix}.$

4. $\begin{vmatrix} 7 & -11 \\ 4 & -6 \end{vmatrix}.$ **5.** $\begin{vmatrix} a & 5 \\ 4a & 2 \end{vmatrix}.$ **6.** $\begin{vmatrix} 2 & 11 \\ 3 & -5 \end{vmatrix}.$

7. $\begin{vmatrix} 3 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 2 & 1 \end{vmatrix}.$ 8. $\begin{vmatrix} 3 & -5 & 1 \\ 4 & -1 & 3 \\ 7 & 0 & 6 \end{vmatrix}.$ 9. $\begin{vmatrix} 1 & -3 & 0 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}.$

10. $\begin{vmatrix} -3 & 5 & 1 \\ -4 & 1 & 3 \\ 3 & 1 & 0 \end{vmatrix}.$ 11. $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}.$ 12. $\begin{vmatrix} 1 & 6 & 4 \\ 1 & 3 & 1 \\ 1 & -1 & -3 \end{vmatrix}.$

13. $\begin{vmatrix} 9 & 6 & 2 \\ 3 & 2 & 3 \\ 6 & 4 & 1 \end{vmatrix}.$ 14. $\begin{vmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 4 & 2 & 1 \end{vmatrix}.$ 15. $\begin{vmatrix} 6 & 7 & 0 \\ 0 & -9 & 2 \\ 4 & 0 & -5 \end{vmatrix}.$

51. Solution by Determinants. By either of the methods of Section 45 the system of equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

may be solved for x and y. In this way we obtain the solution

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

The fractions in the solution involve the coefficients of the unknowns and the constant terms. They could be used as formulas for finding the solution of any pair of linear equations in two unknowns, provided the denominator $a_1b_2 - a_2b_1$ is not zero. But it will be easier to remember the solution if we notice that the numerator and denominator of each fraction are the expansions of determinants of the second order. That is,

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = b_2c_1 - b_1c_2,$$

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1,$$

and

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Hence the solution is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

These formulas apply except when the denominator $a_1b_2 - a_2b_1 = 0$, or $a_1b_2 = a_2b_1$. In this case the left members of the equations can be made the same. Thus, if $a_1b_2 = a_2b_1$, we have

$$\frac{a_2}{a_1}(a_1x + b_1y) = a_2x + \frac{a_2b_1y}{a_1} = a_2x + \frac{a_1b_2y}{a_1} = a_2x + b_2y.$$

The formulas fail, therefore, when the given equations are inconsistent or when they are dependent.

It is quite easy to write the determinants for the solution of a pair of equations. An examination of the determinants involved shows the following statements to be correct:

1. Write the equations so that each unknown of one equation is directly under the same unknown of the other equation, and with the constant terms forming the right members of the system. The coefficients of the unknowns are then arranged as they should appear in the determinant of the denominators, which are the same.

2. The determinant of the numerator of an unknown is formed from the denominator by replacing the coefficients of that unknown by the corresponding constant terms.

Example 1. Solve the system

$$3x + 2y = 3$$
$$2x - 5y = -17.$$

Solution. The equations are in the desired form. Hence we have

$$x = \frac{\begin{vmatrix} 3 & 2 \\ -17 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -5 \end{vmatrix}} = \frac{-15 + 34}{-15 - 4} = \frac{19}{-19} = -1,$$

$$y = \frac{\begin{vmatrix} 3 & 3 \\ 2 & -17 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 2 & -5 \end{vmatrix}} = \frac{-51 - 6}{-15 - 4} = \frac{-57}{-19} = 3.$$

Formulas for the solution of three linear equations in three unknowns may be derived by proceeding in much the same way as we did with a system of two unknowns. By the method of Section 48 it may be shown that the solution of the system

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3},$$

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3},$$

$$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}.$$

The denominators of these fractions are all the same. This common denominator and the numerators are expansions of determinants of third order. By expanding the determinants it may be shown that the solution of the system may be written as

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

These determinant formulas may be used for solving any system of three linear equations in three unknowns if the denominator determinant is not zero. The following statements may be helpful in the use of the formulas:

1. Write the equations with the same unknowns in columns, and with the constant terms forming the right members. The coefficients of the unknowns are then arranged properly for the denominator determinant.

2. The determinant of the numerator of any unknown may be had from the denominator by replacing the coefficients of that unknown by the corresponding constant terms.

Example 2. Solve the system

$$2x + 3y + z = 9$$

$$x - 2y - 3z = 1$$

$$5x + 4y + 6z = 5.$$

Solution. The equations are in proper form for using determinants. Hence we have

$$x = \frac{\begin{array}{|rrr|rr} 9 & 3 & 1 & 9 & 3 \\ 1 & -2 & -3 & 1 & -2 \\ 5 & 4 & 6 & 5 & 4 \end{array}}{\begin{array}{|rrr|rr} 2 & 3 & 1 & 2 & 3 \\ 1 & -2 & -3 & 1 & -2 \\ 5 & 4 & 6 & 5 & 4 \end{array}} = \frac{-108 - 45 + 4 + 10 + 108 - 18}{-24 - 45 + 4 + 10 + 24 - 18} = \frac{-49}{-49} = 1,$$

$$y = \frac{\begin{array}{|rrr|rr} 2 & 9 & 1 & 2 & 9 \\ 1 & 1 & -3 & 1 & 1 \\ 5 & 5 & 6 & 5 & 5 \end{array}}{-49} = \frac{12 - 135 + 5 - 5 + 30 - 54}{-49} = \frac{-147}{-49} = 3,$$

$$z = \frac{\begin{array}{|rrr|rr} 2 & 3 & 9 & 2 & 3 \\ 1 & -2 & 1 & 1 & -2 \\ 5 & 4 & 5 & 5 & 4 \end{array}}{-49} = \frac{-20 + 15 + 36 + 90 - 8 - 15}{-49} = \frac{98}{-49} = -2.$$

Example 3. Solve the system

$$\begin{aligned} x + z - 3 &= 0 \\ 2x - 3y + 10z &= 2 \\ y + z + 6 &= 0. \end{aligned}$$

Solution. A missing unknown can be supplied in an equation by using zero as the coefficient. Thus we write the system in the convenient form

$$\begin{aligned} x + 0y + z &= 3 \\ 2x - 3y + 10z &= 2 \\ 0x + y + z &= -6. \end{aligned}$$

Proceeding as in Example 1, the student should set up the determinants and find the solution $x = 5$, $y = -4$, $z = -2$.

Exercise 31

Solve the problems of Exercise 26, which are consistent, by determinants.
Solve problems **1** to **12** and **23** to **30** of Exercise 28 by determinants.

Chapter Seven

RATIO, PROPORTION, AND VARIATION

52. Ratio. Two numbers may be compared in size by dividing one by the other. The quotient thus obtained is called the **ratio** of one number to the other. If x and y stand for two numbers, the ratio of x to y may be expressed in any of the forms

$$x \div y, \quad x : y, \quad \frac{x}{y}.$$

Two like quantities may be compared in magnitude by first expressing their measures in the same unit of measure. For example, the ratio of 3 days to 18 hours is $72 \div 18 = 3$. Similarly, ratios of the heights of two trees, of the weights of two men, of the speeds of two cars, etc., can be found. Often the magnitudes of unlike quantities are expressed as a ratio. The velocity v of a car which travels d miles in t hours is given by the ratio of d to t; that is,

$$v = \frac{d}{t}.$$

Example 1. In an election the votes for A and B were 600 and 400 respectively. Write the ratio of their votes as a fraction in lowest terms.

Solution. Dividing 600 by 400, we get $\frac{3}{2}$. Hence the ratio of A's votes to B's votes is 3 to 2. The same information is given by saying the ratio of B's votes to A's votes is 2 to 3.

Example 2. Separate 78 into two parts which have the ratio 3 to 5.

Solution. Letting $3x$ and $5x$, which have the ratio 3 to 5, stand for the parts, we have

$$3x + 5x = 78,$$
$$8x = 78,$$
$$x = 9.75.$$

Therefore, $3x = 29.25$ and $5x = 48.75$ are the required parts.

53. Proportion. A statement of equality of two ratios is called a **proportion.** A proportion may be written in any of the forms

$$\frac{a}{b} = \frac{c}{d}, \quad a \div b = c \div d, \quad a:b = c:d.$$

The numbers a and d are called the **extremes,** and b and c are called the **means.** The number d is called the **fourth proportional** to a, b, and c. For the special case in which $b = c$, the proportion becomes

$$\frac{a}{b} = \frac{b}{d}.$$

In this case b is called the **mean proportional** between a and d, and d is the **third proportional** to a and b.

Example 1. Show that in a proportion the product of the extremes is equal to the product of the means.

Solution. Let the proportion be expressed by $a/b = c/d$. Multiplying both sides by bd, we have

$$ad = bc.$$

Example 2. Find the third proportional to 8 and 14.

Solution. Letting x stand for the third proportional, we have

$$\frac{8}{14} = \frac{14}{x},$$

$$8x = 196,$$

$$x = 24\tfrac{1}{2}.$$

Proportions are used in geometry to express relations among parts of similar figures. We mention the following theorems:

1. Corresponding sides of similar figures are proportional.

2. Corresponding areas of similar figures are proportional to the squares of any two corresponding lines.

3. The volumes of similar solids are proportional to the cubes of any two corresponding lines.

Example 3. A triangle has sides of lengths 12, 17, and 22 inches. If the shortest side of a similar triangle is 8 inches, find the longest side. Find also the ratio of the areas.

Solution. Denoting the longest side by x, we may write

$$\frac{x}{22} = \frac{8}{12},$$

$$12x = 176,$$

$$x = 14\tfrac{2}{3} \text{ in.}$$

We indicate the areas by A and a. Then we have the proportion

$$\frac{a}{A} = \frac{8^2}{12^2} = \frac{4}{9}.$$

The ratio of the smaller to the larger area is 4 to 9.

Exercise 32

Express each of the following ratios in fractional form and simplify:

1. 4 ft to 8 in.
2. 30¢ to $2.10.
3. 5 days to 70 hr.
4. 8 qt to 3 gal.
5. 4 lb to 8 oz.
6. 80 min to 2 hr.
7. 40 sq in. to 2 sq ft.
8. 256 cu in. to 1 cu ft.
9. 440 yd to 2 mi.
10. $\frac{27}{7}:\frac{9}{14}$.
11. $\frac{33}{8}:\frac{55}{24}$.
12. $\frac{21}{5}:\frac{14}{10}$.

Find the fourth proportional to each set of numbers:

13. 5, 7, 30.
14. 13, 10, 39.
15. 8, 3, 6.
16. -2, 5, 9.
17. -3, 11, -6.
18. -2, -4, -7.

Find the third proportional to each pair of numbers:

19. 4, 10.
20. 6, 5.
21. 7, 21.
22. 7, 9.
23. 10, 3.
24. 12, 2.

Find the unknown in each of the following proportions:

25. $5:a = 2:18$.
26. $\frac{1}{2}:x = x:18$.
27. $x:49 = 3:7$.
28. $\dfrac{11}{a} = \dfrac{13}{26}$.
29. $\dfrac{4}{x+4} = \dfrac{6}{5}$.
30. $\dfrac{7}{21} = \dfrac{x}{5}$.

31. Divide 72 into two parts which have the ratio 3 to 5.

32. Divide $83.50 into two amounts which have the ratio 3 to 7.

33. Two men, working at the same rate of pay, do a job for $94.50. If one man works 3 days and the other 4 days, find each man's share.

34. Two towns on a certain map are 3 inches apart. If the map scale is 1 inch to 40 miles, find the distance between the two towns.

35. Two school buildings on a city map are 8 inches apart. If the map scale is 5 inches to 2 miles, find the distance between the buildings.

36. A man 6 feet tall casts a shadow 14 feet long. At the same time a flagpole casts a shadow 42 feet long. Find the height of the flagpole.

37. A boy 5 feet tall is 20 feet from a street light. Find the height of the light if the boy's shadow is 12 feet long.

38. The sides of a quadrilateral are 6, 7, 9, and 13 inches. The shortest side of a similar quadrilateral is 4 inches. Find the other sides and the ratio of the areas.

39. The sides of a triangle are 8, 12, and 15 inches. The longest side of a similar triangle is 10 inches. Find the other sides and the ratio of the areas.

40. Compare the area of two circles if their radii have the ratio of 2 to 5.

41. Compare the surface areas and the volumes of the earth and moon if the ratio of the earth's radius to the moon's radius is 11 to 3.

42. Compare the surface areas and volumes of two spheres whose radii have the ratio 2 to 3.

Given $a:b = c:d$, prove the proportions **43** through **46.**

43. $a:c = b:d$. (Proportion by alteration)

44. $(a + b):b = (c + d):d$. (Proportion by addition) Hint: $\frac{a}{b} + 1 = \frac{c}{d} + 1$.

45. $(a - b):b = (c - d):d$. (Proportion by subtraction)

46. $(a + b):(a - b) = (c + d):(c - d)$. (Proportion by addition and subtraction)

54. Direct Variation. If a variable y is always equal to a constant times another variable x, then y is said to **vary directly** as x. The relation may be written as

$$y = kx.$$

The fixed number k is called the **constant of proportionality,** or the **constant of variation.** The expressions

"y varies directly as x,"
"y varies as x,"
"y is directly proportional to x,"
"y is proportional to x,"

are all used with the same meaning.

The circumference of a circle, given by the formula $c = 2\pi r$, varies directly as the radius. The constant of variation is 2π. The distance which a car travels, at uniform speed, is proportional to the time t. The relation is $s = kt$. If s is in miles and t in hours, then k is the speed of the car in miles per hour.

Suppose a variable z is always equal to a constant times the product of two variables x and y; that is,

$$z = kxy.$$

This means that z varies directly as the product of x and y, or is proportional to x and y. The relation is also expressed by saying z **varies jointly** as x and y.

The area of a triangle varies jointly as the base and altitude. The formula $A = \frac{1}{2}bh$ shows the constant of proportionality to be $\frac{1}{2}$.

Example. A constant force moves a body in a line on a smooth surface. The distance which the body moves varies jointly as the force and the square of the time. If the distance is 48 feet when the force is 8 pounds and the time 2 seconds, find the distance when the force is 6 pounds and the time is 3 seconds.

Solution. Let s stand for the distance, f for the force, and t for the time. Then we write

$$s = kft^2.$$

To find k, we substitute 48 for s, 8 for f, and 2 for t. Thus we have

$$48 = k(8)(4) \quad \text{and} \quad k = \tfrac{3}{2}.$$

The relation may now be written as

$$s = \tfrac{3}{2}ft^2.$$

This formula will give the distance for any force and any time. Substituting the given values $f = 6$ and $t = 3$, we obtain

$$s = \tfrac{3}{2}(6)(9) = 81 \text{ feet.}$$

55. Inverse Variation. If a variable y is always equal to a constant times the reciprocal of another variable x, then y is said to **vary inversely** as x, or to be **inversely proportional** to x. Hence

$$y = \frac{k}{x}.$$

The time required for water to fill a tank is inversely proportional to the rate at which water enters the tank. And the volume of a confined gas varies inversely as the pressure, the temperature of the gas remaining the same.

Example 1. The volume of gas in a container is 800 cubic inches at a pressure of 12 pounds per square inch. Find its volume at a pressure of 21 pounds per square inch.

Solution. The volume being inversely proportional to the pressure, we have

$$v = \frac{k}{p}.$$

Since $v = 800$ when $p = 12$, we obtain

$$800 = \frac{k}{12},$$

and

$$k = 9600.$$

The formula may be written now as

$$v = \frac{9600}{p}.$$

Substituting 21 for p, we find

$$v = \tfrac{9600}{21} = 457\tfrac{1}{7} \text{ cubic inches.}$$

The different kinds of variation may occur simultaneously in problems. In the relation

$$z = \frac{kx}{y},$$

z varies directly as x and inversely as y. The relation

$$W = \frac{kxy^3}{z^2},$$

means W varies jointly as x and y^3 and inversely as z^2.

Example 2. If z is directly proportional to x and inversely proportional to y, and if $z = 10$ when $x = 4$ and $y = 6$, find the constant of variation. Then evaluate z when $x = 9$ and $y = 8$.

Solution.

$$z = \frac{kx}{y}.$$

Substituting $z = 10$, $x = 4$, $y = 6$, we find $k = 15$; hence for all values of x and y

$$z = \frac{15x}{y}.$$

For $x = 9$ and $y = 8$,

$$z = \frac{15(9)}{8} = \frac{135}{8} = 16\tfrac{7}{8}.$$

Example 3. The maximum safe load for a beam varies jointly as the breadth and square of its depth and inversely as its length. If a beam 12 feet long, 4 inches wide, and 8 inches deep, can safely bear a load up to 2000 pounds, find the maximum safe load for a beam of the same material which is 10 feet long, 2 inches wide, and 6 inches deep.

Solution. Using the letters w, l, b, and d, respectively, for the load, length, breadth, and depth we write the law for the beam as

$$w = \frac{kbd^2}{l}.$$

To find k, we substitute 2000, 4, 8, and 12, respectively, for w, b, d, and l and obtain

$$2000 = \frac{k(4)(8^2)}{12},$$

$$k = \frac{375}{4}.$$

The formula may be written

$$w = \frac{375bd^2}{4l},$$

where b and d are in inches, l in feet, and w in pounds. Hence for $b = 2$, $d = 6$, and $l = 10$, we have

$$w = \frac{375(2)(6^2)}{4(10)} = 675 \text{ pounds.}$$

Exercise 33

Express the statements **1** through **7** as equations, using k for the constant of variation.

1. The surface area S of a sphere varies as the square of the radius r.

2. The area A of a rectangle varies jointly as the base b and the altitude h.

3. The volume v of a circular cylinder is proportional to its altitude h and the square of its radius r.

4. W is directly proportional to x^3 and inversely proportional to y^2.

5. The attraction A between two bodies varies inversely as the square of the distance d between them.

6. W is directly proportional to xy and inversely proportional to z^2.

7. V varies as x and inversely as y^2z.

Write the relation among the variables in problems **8** through **12**.

8. Z varies jointly as x and y, and $Z = 8$ when $x = 10$ and $y = 4$.

9. A varies as r^2, and $A = \pi$ when $r = 1$.

10. V is proportional to r^2 and h, and $V = 8\pi$ when $r = 2$ and $h = 2$.

11. W varies inversely as x, and $W = 12$ when $x = 5$.

12. W varies directly as x^2y and inversely as z, and $W = 11$ when $x = 3$, $y = 1$, and $z = 5$.

13. A varies directly as x^2, and $A = 15$ when $x = 5$. Find A when $x = 7$.

14. P is inversely proportional to x, and $P = 9$ when $x = 4$. Find P when $x = 12$.

15. W is proportional to xy^2, and $W = 1000$ when $x = 3$ and $y = 10$. Find W when $x = 6$ and $y = 5$.

16. Z varies directly as xy and inversely as w^2, and $Z = 1200$ when $x = 4$, $y = 9$, and $w = 6$. Find Z when $x = 3$, $y = 12$, and $w = 9$.

17. W varies as xy^3 and inversely as z^2, and $W = 20$ when $x = 2$, $y = 3$, and $z = 5$. Find W when $x = 8$, $y = 6$, and $z = 10$.

18. The volume of a confined gas is inversely proportional to the pressure. If the volume is 1000 cubic inches when the pressure is 12 pounds per square inch, find the volume when the pressure is (a) 8 pounds; (b) 16 pounds; (c) 20 pounds.

19. The distance traveled by a falling body is proportional to the square of the time it spends in falling. If a body falls 64 feet in 2 seconds, find the distance which it will fall in (a) 3 seconds; (b) 4 seconds; (c) 10 seconds.

20. The maximum safe load for a beam 14 feet long, 4 inches wide, and 5 inches deep is 3000 pounds. Find the maximum safe load for a beam of the same material which is 12 feet long, 5 inches wide, and 9 inches deep. (See illustrative Example 3, Section 55.)

21. Compare the maximum safe load for a beam 10 feet long, 6 inches wide, and 10 inches deep with the maximum safe load for the same beam when it is turned so that the 6-inch dimension is the depth and the 10-inch dimension is the width.

22. The weight of a body above the earth's surface varies inversely as the square of the distance from the center of the earth. A body weighing 100 pounds at the surface (4000 miles from the center) would have what weight at a distance above the surface of (a) 1000 miles; (b) 4000 miles?

23. The illumination on a page is directly proportional to the wattage of the lamp and inversely proportional to the square of the distance from the lamp. If the distance is increased from 2 feet to 4 feet and the lamp wattage is doubled, by what percent is the illumination changed?

24. If one holds his paper 2 feet from a 50-watt lamp, how far should the paper be held from a 200-watt lamp to receive equal illumination? (See problem **23.**)

25. A crew of 12 men working 7 hours per day do a job in 15 days. How long will it take 9 men working 10 hours per day to do the job?

26. The time required for a pendulum to make one vibration (a round trip) is proportional to the square root of its length. If a pendulum 30 inches long makes one vibration in 1.92 seconds, find the vibration period of a pendulum of length (a) 25 inches; (b) 100 inches; (c) 4 inches.

27. Find the length of a pendulum which has a vibration period of (a) 1 second; (b) 2 seconds. (See problem **26.**)

28. The electrical resistance of a wire varies directly as the length and inversely as the square of its radius. The resistance of a certain piece of wire of radius 0.02 inch and 1000 feet long is 300 ohms. Find the resistance of a wire of the same material 1500 feet long and 0.03 inch in radius.

29. Compare the resistances of two pieces of wire of the same material if one piece is 100 feet long with radius 0.01 inch and the other is 160 feet long with radius 0.04 inch. (See problem **28.**)

30. The volume of a confined gas varies directly as its absolute temperature and inversely as the pressure. If the volume is 800 cubic inches when the absolute temperature is 300° and the pressure is 20 pounds per square inch, what will be the volume when the absolute temperature is 200° and the pressure 40 pounds per square inch?

31. The absolute temperature of a gas is doubled and the volume is halved. Find the ratio of the original pressure to the final pressure. (See problem **30.**)

Chapter Eight

EXPONENTS AND RADICALS

56. Laws of Exponents. In Section 7 the definition of a positive integral exponent was given. Accordingly, a^n (n a positive integer) means the product obtained by taking a as a factor n times. That is,

$$a^n = a \cdot a \cdot a \cdot \cdots \cdot a. \quad (n \text{ factors})$$

In this chapter we shall consider other powers than the positive integral powers of numbers. The new powers will be introduced by extending the definition of exponents to include all rational numbers (Sec. 2). Before taking up new exponents, however, we shall state five laws of exponents, and prove that the laws hold for positive integral exponents.

Law I.

$$a^m a^n = a^{m+n}.$$

Proof: a^m means the product of m factors, each of which is a, and a^n means that a is used as a factor n times. Hence the product of these two products has a as a factor $m + n$ times, which by definition is equal to a^{m+n}.

Law II.

$$\frac{a^m}{a^n} = a^{m-n}, \quad \text{if} \quad m > n \quad \text{and} \quad a \neq 0. \quad (>, \text{ read "is greater than.")}$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad \text{if} \quad n > m \quad \text{and} \quad a \neq 0. \quad (\neq, \text{ read "is not equal to.")}$$

Proof: We specify that a is a number other than zero because zero is not used as a divisor. This law is easily established by the process of reducing a fraction to lowest terms. Thus, by dividing, if $m > n$, each of the n factors of the denominator may be replaced by 1 and n factors of the numerator become 1. This makes the denominator equal to 1, and the numerator, besides the unit factors, has a left as a factor $m - n$ times. Hence the quotient is a^{m-n}. Similarly, if $n > m$, the numerator becomes 1 and the denominator a^{n-m}.

LAW III.

$$(a^m)^n = a^{mn}.$$

Proof: $(a^m)^n$ means a^m is to be taken as a factor n times. But each a^m has a as a factor m times. Hence, in all, a will appear as a factor mn times, giving a^{mn}.

LAW IV.

$$(ab)^m = a^m b^m.$$

Proof: $(ab)^m$ means the product obtained by taking ab as a factor m times. The factor a then occurs m times, yielding a^m; and the factor b occurs m times, giving b^m. Hence, $(ab)^m = a^m b^m$.

LAW V.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad \text{if} \quad b \neq 0.$$

Proof: $(a/b)^m$ means the product of m factors each of which is a/b. The numerator of the product is the product of the numerators, and is equal to a^m. Similarly, the denominator of the product is b^m.

Illustrations:

$$2^2 \cdot 2^3 = 2^5 \qquad (2^2)^3 = 2^6 \qquad (2a^2)^4 = 2^4 a^8$$

$$\frac{6^5}{3^5} = 2^5 \qquad \frac{a^{2n+1}}{a^n} = a^{n+1} \qquad \frac{a^{3n+2}b^m}{a^{2n+1}b^{m+3}} = \frac{a^{n+1}}{b^3}$$

$$\left(\frac{x^2y}{xy^3}\right)^3 \left(\frac{x^2y^4}{x^5y}\right)^2 = \left(\frac{x}{y^2}\right)^3 \left(\frac{y^3}{x^3}\right)^2 = \frac{x^3}{y^6} \cdot \frac{y^6}{x^6} = \frac{1}{x^3}$$

$$\frac{(3^2x^2y^3)^4}{(3x^3y)^4} = \left(\frac{3^2x^2y^3}{3x^3y}\right)^4 = \left(\frac{3y^2}{x}\right)^4 = \frac{81y^8}{x^4}$$

Exercise 34

Perform the indicated operations, using the laws of exponents:

1. $3^2 \cdot 3^3$.
2. $4^4 \cdot 4^2$.
3. $a^m a^3$.
4. $(3^2)^3$.
5. $(4^n)^2$.
6. $(a^m)^2$.
7. $(2a^3)(3a^2)$.
8. $(3x^2)(x^3y)(4y)$.
9. $(-a^3)(-a^2)a^4$.

10. $(3^2ab^2)(2a^3b)$.
11. $(-3a)^2(-3a)a^2$.
12. $(-a^2)(-b)^3a^2b^3$.
13. $(2x^2y^3)^3$.
14. $(3ab^3)^2$.
15. $(2xy^2)^4$.
16. $(3c^2)^2(2c^3)^2$.
17. $(x^2y)^3(xy^2)^3$.
18. $(3ab)(a^2b^3)^2$.
19. $\left(\frac{a^2}{3}\right)^4\left(\frac{6}{a}\right)^4$.
20. $\left(\frac{4a}{b^2}\right)^3\left(\frac{b}{2a^2}\right)^3$.
21. $\left(\frac{8}{ab^2}\right)^5\left(\frac{a^2b}{4}\right)^5$.
22. $\frac{2x^5y^2}{3xy^3}$.
23. $\frac{3a^4b^6}{6ab^7}$.
24. $\frac{5x^4b}{15xb^5}$.
25. $\frac{4x^3y^2z^4}{2^2xy^3z^3}$.
26. $\frac{3^2x^6y^6z}{18xyz^6}$.
27. $\frac{3^3a^4b^3c}{2^2a^2bc^3}$.
28. $\left(\frac{3x^4y^7}{2xy^5}\right)^3$.
29. $\left(\frac{4a^3b^2c}{2^3a^2bc^3}\right)^4$.
30. $\left(\frac{5^2x^4y^3z^2}{15x^5yz^3}\right)^2$.
31. $\frac{(p^3qr^2)^5}{(pq^2r)^5}$.
32. $\frac{(3p^4q^3r^2)^4}{(6p^2q^4r^3)^4}$.
33. $\frac{(6^2p^5r^4)^3}{(3^3p^7q^5r^3)^3}$.
34. $\frac{(3x^3y^3)^3}{(6xy^6)^2}$.
35. $\frac{(5x^4y^3)^2}{(10x^5y^6)^3}$.
36. $\frac{(8xy^4)^3}{(4x^4y)^4}$.
37. $\frac{3^2ab^3}{2^3c^5} \cdot \frac{8a^3c^3}{9b^3}$.
38. $\frac{4x^3y^3}{10z^4} \cdot \frac{5xz^5}{8y^2}$.
39. $\frac{9pq^3}{2r^4} \cdot \frac{4p^2r^2}{6p^3}$.
40. $\frac{x^{3n}}{x^{2n}}$.
41. $\frac{2^{2n+1}}{2^n}$.
42. $\frac{3^{4n+1}}{3^{4n}}$.
43. $\frac{5^{2n}x^{m+2}}{5^nx^m}$.
44. $\frac{2^na^{n+1}}{2^{3n}a^n}$.
45. $\frac{a^{n+2}b^{n+1}}{a^nb^n}$.
46. $\left(\frac{3x}{4y}\right)^4\left(\frac{2y}{3x^3}\right)^5$.
47. $\left(\frac{x^3y^2}{a^2b^3}\right)^3\left(\frac{ab^2}{xy^4}\right)^4$.
48. $\left(\frac{2ab^4}{x^4y}\right)^3\left(\frac{xy^4}{a^4b}\right)^2$.
49. $\left(\frac{x^6y^9z^7}{x^4z^2}\right)^2\left(\frac{xyz}{x^3yz^2}\right)^4$.
50. $\left(\frac{2x^4y^3}{3y^2z^5}\right)^4\left(\frac{6y^2z^2}{4x^2y^5}\right)^3$.
51. $\left(\frac{a^3b^2c}{b^3c^4}\right)^5\left(\frac{b^4c^5}{a^2b^2c^2}\right)^4$.

57. Zero and Negative Integral Exponents. We have defined positive integral exponents and have established five laws of operations which apply to them. Our next step is to extend the idea of exponents to include zero and the negative integers. The new exponents are to be defined in such a way that they obey the five laws of exponents.

First, let us determine what meaning should be given to a number whose exponent is zero. If Law II is to hold when $m = n$, we have

$$\frac{a^n}{a^n} = a^{n-n} = a^0 \quad (a \neq 0).$$

This division yields a zero exponent. But any nonzero number divided by itself has 1 as the quotient. We shall therefore assign the value 1 to any nonzero number which has zero for its exponent. That is, by definition,

$$\mathbf{a^0 = 1.}$$

Now, in a similar way, we may determine what meaning is to be given to a negative integral exponent. If Law I is to hold when $m = -n$, then

$$a^{-n}a^n = a^0 = 1.$$

Dividing both members of $a^{-n}a^n = 1$ by a^n, we have

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0).$$

We thus define a negative integral exponent in terms of a positive exponent.

The definitions of zero and negative integral exponents were reached, in each case, by applying one of the laws of exponents. However, it would be easy to show that they conform to each of the five laws. We note also that the definitions of exponents thus far place no restriction on the base a except that $a \neq 0$ when the exponent is zero or a negative integer.

Illustrations:

$$9^0 = 1 \qquad (-9)^0 = 1 \qquad a^5a^{-3} = a^{5-3} = a^2$$

$$(2^{-3})^{-2} = 2^6 \qquad (2^{-3})^2 = 2^{-6} = \frac{1}{2^6} \qquad \frac{x^4}{x^{-2}} = x^{4-(-2)} = x^6$$

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{(\frac{2}{3})^3} = \left(\frac{3}{2}\right)^3 \qquad \frac{x^{-2}y^3}{a^2b^{-5}} = \frac{\frac{1}{x^2}\cdot y^3}{a^2\cdot\frac{1}{b^5}} = \frac{y^3}{x^2}\cdot\frac{b^5}{a^2} = \frac{b^5y^3}{a^2x^2}$$

Comparing the original fraction and the resulting fraction in each of the last two illustrations, we observe the following statements to be true:

1. *A fraction whose exponent is a negative integer is equal to the reciprocal of the fraction with the sign of the exponent changed.*
2. *A factor of the numerator of a fraction may be moved to become a factor of the denominator, or vice versa, if the sign of the exponent of the factor is changed.*

We write at once, for example,

$$\frac{3^{-1}a^{-3}b^{-2}x^3}{2^{-2}c^{-4}y} = \frac{2^2c^4x^3}{3a^3b^2y}.$$

Terms, however, cannot be thus transferred from one member of a fraction to the other. For example,

$$\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}} \text{ is not equal to } \frac{x+y}{x^2+y^2}.$$

But

$$\frac{x^{-2}+y^{-2}}{x^{-1}+y^{-1}} = \frac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}} = \frac{\frac{y^2+x^2}{x^2y^2}}{\frac{y+x}{xy}} = \frac{y^2+x^2}{x^2y^2}\cdot\frac{xy}{y+x} = \frac{x^2+y^2}{xy(x+y)}.$$

Exercise 35

Find the value of each expression:

1. 4^{-2}.
2. $(-2)^0$.
3. 10^0.
4. 2^{-1}.
5. $(-\frac{1}{2})^0$.
6. 3^{-4}.
7. $5^{-2}\cdot 5^3$.
8. $6^{-4}\cdot 6^5$.
9. $7^3\cdot 7^{-3}$.
10. $4^{-5}\cdot 4^4$.
11. $2^{-3}\cdot 2^{-1}$.
12. $5^2\cdot 5^{-4}$.
13. $\dfrac{2^{-3}\cdot 3^{-1}}{2^2\cdot 3^{-4}}$.
14. $\dfrac{9^{-2}\cdot 9^{-3}}{3^{-9}\cdot 3}$.
15. $\dfrac{6^{-3}\cdot 3^{-3}}{6^{-1}\cdot 3^{-4}}$.
16. $\dfrac{(-2)^{-3}(-2)^2}{(-2)^{-4}}$.
17. $\dfrac{(-10)^{-3}}{(-5)^{-3}}$.
18. $\dfrac{(-3)^{-1}(-3)^{-2}}{(-3)^{-6}}$.
19. $(\frac{2}{9})^{-1}$.
20. $(\frac{1}{5})^{-2}$.
21. $(\frac{5}{3})^{-1}$.
22. $(2^{-1})^3$.
23. $(5^{-2})^{-1}$.
24. $(3^{-2})^{-2}$.
25. $(5^{-1}\cdot 2^3)^{-2}$.
26. $(4^2\cdot 3^{-2})^{-1}$.
27. $(3^{-3}\cdot 6^2)^{-2}$.

Express each of the following without zero or negative exponents and simplify:

28. $x^0y^3y^{-4}$.
29. $x^2x^{-2}y^{-3}$.
30. $x^{-4}y^3y^{-3}$.
31. $ab^{-1}c^{-2}$.
32. $a^2b^0c^{-1}$.
33. $a^3b^{-2}c^0$.
34. $\dfrac{x^2y^{-2}}{x^{-1}y^{-3}}$.
35. $\dfrac{3^{-1}x^{-3}y^2}{2^{-1}x^{-5}y}$.
36. $\dfrac{21x^0y^{-5}}{7^2x^{-1}y^{-4}}$.
37. $\dfrac{3^2p^{-3}q^{-4}}{2^{-1}p^{-4}q^{-5}}$.
38. $\dfrac{5^{-2}p^2q^3}{10^{-1}p^0q^{-2}}$.
39. $\dfrac{6^{-2}p^{-2}q^{-2}}{3^{-2}p^{-3}q^4}$.
40. $\left(\dfrac{3a}{b^2}\right)^{-1}$.
41. $\left(\dfrac{2a^2b}{3c^3}\right)^{-2}$.
42. $\left(\dfrac{a^3b^0}{2c}\right)^{-1}$.
43. $\left(\dfrac{p^{-2}q^{-3}}{2^0r^{-1}}\right)^{-2}$.
44. $\left(\dfrac{3^2p^{-1}q^{-3}}{6q^0r^{-4}}\right)^{-1}$.
45. $\left(\dfrac{p^{-2}q^{-4}r^5}{p^{-3}q^{-2}r^3}\right)^{-1}$.
46. $\left(\dfrac{a^{-2}b^3c^{-2}}{b^{-1}c^{-2}}\right)^2$.
47. $\left(\dfrac{a^{-1}x^{-2}y^{-2}}{b^{-3}x^{-2}y^3}\right)^3$.
48. $\left(\dfrac{x^{-1}y^2z^{-3}}{y^{-1}z^{-2}}\right)^4$.
49. $\dfrac{2^{-3}+3^{-2}}{3^{-1}}$.
50. $\dfrac{2^{-2}+1^{-2}}{2^{-1}+1^{-1}}$.
51. $\dfrac{3^{-1}-5^{-1}}{3^{-1}+5^{-1}}$.
52. $\dfrac{x^{-2}-y^{-2}}{x^{-2}y^{-2}}$.
53. $\dfrac{x^{-3}y^{-3}}{x^{-3}+y^{-3}}$.
54. $\dfrac{x^{-1}+y^{-2}}{y^{-1}}$.

58. Roots of Numbers. To aid in the study of fractional exponents let us discuss roots of numbers. If a number is equal to the product of two equal factors, one of the factors is called a **square root** of the number. If a number is equal to the product of three equal factors, one of the factors is called a **cube root** of the number. We have the following general definition.

If a and b are two numbers such that the n*th power of a (n a positive integer) is equal to b, then a is called an* **nth root** *of b. That is, if* $a^n = b$, *then a is an* n*th root of b.*

Illustrations of roots of numbers are given by the equations

$$2^2 = 4, \quad (-2)^2 = 4, \quad 3^3 = 27, \quad (-3)^3 = -27.$$

We see that $+2$ and -2 are square roots of 4, that 3 is a cube root of 27, and that -3 is a cube root of -27. In Chapter Sixteen we shall find that any nonzero number has two square roots, three cube roots, four fourth roots, and so on. But some of these roots involve a new kind of number. The new number is different from the real numbers which, we recall, are the positive and negative numbers and zero.

The preceding illustrations of roots are indicative of certain facts which we point out now and consider further in Chapter Sixteen.

1. A positive number has two real nth roots if n is an even positive integer. The roots have opposite signs but equal numerical values.
2. A positive number has one positive nth root and a negative number has one negative nth root if n is an odd positive integer.
3. A negative number has no even root which is a real number because even powers of both positive and negative numbers are positive.

We wish now to give our attention to real roots of real numbers. Hence we defer consideration of numbers and roots of numbers which are not real.

If a number has one real nth root, that root is called the **principal nth root.** If a number has two real nth roots, the positive root is designated as the principal nth root. The symbol $\sqrt[n]{a}$, called a **radical,** is used to stand for the principal nth root of a. The number a is called the **radicand,** and n is the **index,** or **order,** of the radical. The symbol $\sqrt{\ }$ is called the **radical sign.** The following are illustrations of principal roots of numbers.

$$\sqrt{9} = 3, \quad \sqrt[3]{8} = 2, \quad \sqrt[4]{16} = 2, \quad \sqrt[5]{-32} = -2.$$

We note that -3 is a square root of 9 and -2 is a fourth root of 16, but neither is a principal root. The negative of a principal root may be indicated by placing a minus sign before the radical. Thus, $-\sqrt[4]{16} = -2$. It is customary to omit writing the index when a square root is meant.

59. Fractional Exponents. We are now ready to extend the definition of exponents to include fractions of the form m/n where m is a positive or negative integer and n is a positive integer.

First we take $m = 1$ and seek an interpretation of $a^{1/n}$. If Law III is to hold, we have

$$(a^{1/n})^n = a^{n/n} = a.$$

This shows that the nth power of $a^{1/n}$ is a, or that $a^{1/n}$ is an nth root of a. Regarding this root as the principal root, we have, by definition,

$$\boldsymbol{a^{1/n} = \sqrt[n]{a}.}$$

In this definition a may have any positive or negative value when n is odd. When n is even a is restricted to positive values.

Illustrations:

$$36^{1/2} = 6; \quad (-8)^{1/3} = -2; \quad a^{1/4} = \sqrt[4]{a}, \quad a > 0.$$

Applying Law III again, with the integer $m \neq 1$, we have

$$a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m},$$

and also

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a}\,)^m.$$

We define $a^{m/n}$ by these equations. The form $\sqrt[n]{a^m}$ means the principal nth root of a^m, and the form $(\sqrt[n]{a}\,)^m$ means the mth power of the principal nth root of a, the two being equal. In each form the denominator n of the exponent indicates a root and the numerator m indicates a power. We note again, however, that values of a which have no real nth root are excluded in this discussion.

We have now defined exponents to include all rational numbers. We first defined the positive integral exponents and proved the five laws of operations. Then definitions were given for the other exponents in conformity with those laws. Although complete proofs were made only for the positive integral exponents, it would be easy to show that the laws of exponents apply to rational numbers generally.

Illustrations:

$$8^{5/3} = (\sqrt[3]{8}\,)^5 = 2^5 \qquad 8^{-5/3} = \frac{1}{8^{5/3}} = \frac{1}{2^5}$$

$$(-32)^{4/5} = (\sqrt[5]{-32}\,)^4 = (-2)^4 = 16$$

$$(-32)^{3/5} = (\sqrt[5]{-32}\,)^3 = (-2)^3 = -8$$

Observe that in these illustrations it is easier to take the root and then raise the root to the required power than to reverse the order of operating.

Example 1. Perform the multiplication $3x^{5/3}y^{-3/4}\cdot 2x^{-2/3}y^{5/4}$.

Solution.

$$\begin{aligned} 3x^{5/3}y^{-3/4}\cdot 2x^{-2/3}y^{5/4} &= 6x^{5/3-2/3}y^{-3/4+5/4} \\ &= 6xy^{1/2}. \end{aligned}$$

Example 2. Perform the operations indicated by $\dfrac{8x^{2/3}y^0z^{-1/2}}{4x^{1/2}y^{-1/3}z}$.

Solution.

$$\begin{aligned} \frac{8x^{2/3}y^0z^{-1/2}}{4x^{1/2}y^{-1/3}z} &= \frac{2x^{2/3-1/2}y^{1/3}}{z^{1+1/2}} \\ &= \frac{2x^{1/6}y^{1/3}}{z^{3/2}}. \end{aligned}$$

Example 3. Perform the operations indicated by $\left(\dfrac{a^{4/5}b^{-2}c^{3/4}}{ab^{1/3}c^{-5/4}}\right)^{-1/2}$ and leave the result with positive exponents.

Solution.

$$\begin{aligned}\left(\frac{a^{4/5}b^{-2}c^{3/4}}{ab^{1/3}c^{-5/4}}\right)^{-1/2} &= \left(\frac{c^2}{a^{1/5}b^{7/3}}\right)^{-1/2}\\ &= \left(\frac{a^{1/5}b^{7/3}}{c^2}\right)^{1/2}\\ &= \frac{a^{1/10}b^{7/6}}{c}.\end{aligned}$$

Example 4. Expand $(2x^{3/2} + y^{3/2})^2$.

Solution.

$$\begin{aligned}(2x^{3/2} + y^{3/2})^2 &= (2x^{3/2})^2 + 2(2x^{3/2}y^{3/2}) + (y^{3/2})^2\\ &= 4x^3 + 4x^{3/2}y^{3/2} + y^3.\end{aligned}$$

Exercise 36

Find the value of each expression, using the principal root:

1. $25^{1/2}$.
2. $64^{1/3}$.
3. $81^{1/4}$.
4. $8^{2/3}$.
5. $27^{4/3}$.
6. $9^{3/2}$.
7. $(-8)^{5/3}$.
8. $(-125)^{1/3}$.
9. $(-64)^{2/3}$.
10. $8^{-4/3}$.
11. $16^{-3/4}$.
12. $32^{-3/5}$.
13. $(\frac{4}{9})^{5/2}$.
14. $(\frac{4}{9})^{-1/2}$.
15. $(\frac{8}{27})^{-2/3}$.
16. $(0.04)^{-1/2}$.
17. $(0.001)^{2/3}$.
18. $(0.008)^{2/3}$.
19. $(\frac{100}{81})^{-1/2}$.
20. $(-\frac{243}{32})^{-4/5}$.
21. $(-\frac{64}{27})^{-4/3}$.

Simplify by performing the indicated operations and leaving each result without zero or negative exponents:

22. $x^{1/3}x^{5/3}$.
23. $a^{1/2}a^{1/3}$.
24. $c^{1/4}c^{1/6}$.
25. $x^{-3/2}x^{1/4}$.
26. $a^{-4/5}a^{3/5}$.
27. $c^{1/4}c^{-3/4}$.
28. $\dfrac{x^{-1/4}}{x^{1/3}}$.
29. $\dfrac{a^{5/2}}{a^{-1/3}}$.
30. $\dfrac{c^{1/4}}{c^{-3/2}}$.
31. $(5a^{-2/3})^3$.
32. $(2c^{-1/2})^{-3}$.
33. $(3x^{-1/4})^{-4}$.
34. $(2a^2b^0)^{1/2}$.
35. $(a^3b^2)^{-1/2}$.
36. $(a^3b^3)^{-2/3}$.
37. $(5x^{-2/3})^{-3}$.
38. $(2y^{-7/8})^{-4}$.
39. $(x^0y^{-3/4})^{-4}$.
40. $4x^{3/4}y^{-1/3}\cdot 3x^{-5/6}y^{2/3}$.
41. $5a^{-2}b^{-1/3}\cdot 2a^{3/4}b^{5/3}$.
42. $ab^2c^{1/2}\cdot 2a^{-1}b^{2/3}$.
43. $6a^{-1/2}b^{1/3}\cdot 2a^{-1/3}b^{-1/2}$.
44. $2x^{-1/4}y^{2/3}\cdot 4x^{5/4}y^{5/6}$.
45. $5a^{-5/2}b^{3/4}\cdot 2a^{3/2}b^{2/3}$.
46. $\dfrac{4^{1/2}x^{1/2}y^{2/3}}{8^{1/3}x^{-1/2}y^{1/3}}$.
47. $\dfrac{9^{3/2}a^{-1/2}b^{3/2}}{4^{3/2}a^{1/2}b^{-1/2}}$.
48. $\dfrac{27^{1/3}a^{1/4}b^{1/3}}{16^{1/2}a^{-3/4}b^{1/2}}$.

49. $\left(\dfrac{x^0y^{-2/5}}{x^{1/2}y^{-3/5}}\right)^{-5}$.
50. $\left(\dfrac{x^{-3}y^{-1/2}}{x^{1/3}y^{2/3}}\right)^{-3}$.
51. $\left(\dfrac{x^{-5/4}y^{4/5}}{x^{-1/5}y^{-1/4}}\right)^{-4}$.
52. $\left(\dfrac{16a^{-4}b^{-3/2}}{9a^{-2}b^{1/3}}\right)^{-1/2}$.
53. $\left(\dfrac{8a^{-1/2}b^{3/4}}{27ab^{-1/2}}\right)^{2/3}$.
54. $\left(\dfrac{4a^{-3}b^{4/5}}{25a^{1/5}b^0}\right)^{-3/2}$.
55. $(x^{1/2} + 1)(x^{1/2} - 1)$.
56. $(x^{1/2} - y^{1/2})^2$.
57. $(x^{2/3} + 3)(x^{2/3} - 3)$.
58. $(2x^{1/2} - 3y^{1/2})(x^{1/2} + 4y^{1/2})$.
59. $(x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$.
60. $(2x^{3/2} + 3y^{3/2})(x^{3/2} - 2y^{3/2})$.
61. $(x^{1/3} + y^{1/3})(x^{2/3} - x^{1/3}y^{1/3} + y^{2/3})$.

60. Equivalent Radical Forms. From the laws of exponents certain useful laws of radicals may be had. We write here four laws of radicals. These four relations are obtained by writing the exponential relations, appearing to the right, in terms of radicals. We recall that a and b may take only the values for which the indicated roots are principal roots.

$$\sqrt[n]{a^n} = (\sqrt[n]{a}\,)^n = a \qquad (a^n)^{1/n} = (a^{1/n})^n = a$$

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \qquad a^{1/n}b^{1/n} = (ab)^{1/n}$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} \qquad (a^{1/m})^{1/n} = a^{1/mn}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \qquad \frac{a^{1/n}}{b^{1/n}} = \left(\frac{a}{b}\right)^{1/n}$$

These four relations may be employed in making some of the needed changes in radicals, the most common of which are:

A. *Removing factors from the radicand.*
B. *Making the radicand non-fractional.*
C. *Expressing a radical as one of lower order.*
D. *Bringing an outside factor inside the radical sign.*

A radical is said to be in **simplest form** when the operations **A, B,** and **C** are fully carried out. The operation **B** is called **rationalizing the denominator.**

A. If a factor of the radicand of an nth order radical is a perfect nth power, the factor may be removed by writing its principal nth root as a coefficient of the radical. This operation may be readily performed by first factoring the radicand.

Illustrations:

$$\sqrt{75x^3} = \sqrt{25x^2(3x)} = \sqrt{25x^2}\sqrt{3x} = 5x\sqrt{3x}$$

$$\sqrt[3]{108(x+y)^7} = \sqrt[3]{3^3\cdot 4(x+y)^6(x+y)} = 3(x+y)^2\sqrt[3]{4(x+y)}$$

B. A fractional radicand of an *n*th order radical may be made nonfractional by multiplying the numerator and the denominator of the fraction by an expression which makes the denominator a perfect *n*th power, and then taking the *n*th root of the denominator.

Illustrations:

$$\sqrt{\frac{2}{5}} = \sqrt{\frac{5}{5}\cdot\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{\sqrt{25}} = \frac{\sqrt{10}}{5} = \frac{1}{5}\sqrt{10}$$

$$\sqrt[4]{\frac{ab}{8c}} = \sqrt[4]{\frac{2c^3}{2c^3}\cdot\frac{ab}{8c}} = \sqrt[4]{\frac{2abc^3}{16c^4}} = \frac{\sqrt[4]{2abc^3}}{2c} = \frac{1}{2c}\sqrt[4]{2abc^3}$$

C. If the radicand of an *n*th order radical is a perfect *m*th power, and if *m* and *n* have a common factor greater than 1, then the order of the radical can be reduced by removing the common factor from *m* and *n*.

Illustrations:

$$\sqrt[4]{25a^2} = \sqrt[4]{(5a)^2} = (5a)^{2/4} = (5a)^{1/2} = \sqrt{5a}$$

$$\sqrt[8]{5^2a^4b^6} = 5^{2/8}a^{4/8}b^{6/8} = 5^{1/4}a^{2/4}b^{3/4} = \sqrt[4]{5a^2b^3}$$

$$\sqrt[6]{324(x^2 - 2xy + y^2)} = \sqrt[6]{2^2 3^4 (x - y)^2} = \sqrt[3]{2\cdot 3^2(x - y)} = \sqrt[3]{18(x - y)}$$

In each of the first two of the three preceding illustrations we expressed a radical in terms of fractional exponents, reduced the exponents to fractions in lower terms, and then changed back to a radical. Observe that these intermediate steps may be omitted. The result is achieved more directly, as shown in the last illustration, by dividing the index and the exponents of the radicand by their greatest common factor.

D. The coefficient or a factor of the coefficient of an *n*th order radical may be raised to the *n*th power and written as a factor of the radicand.

Illustrations:

$$2\sqrt[3]{5x} = \sqrt[3]{2^3}\sqrt[3]{5x} = \sqrt[3]{2^3(5x)} = \sqrt[3]{40x}$$

$$2x\sqrt{1 - \frac{1}{4x^2}} = \sqrt{4x^2\left(1 - \frac{1}{4x^2}\right)} = \sqrt{4x^2 - 1}$$

Exercise 37

Write each of the following expressions as a radical:

1. $a^{2/3}$.
2. $5^{1/2}$.
3. $x^{1/2}y^{1/3}$.
4. $3a^{1/5}$.
5. $2^{1/2}x^{3/2}$.
6. $(4y^2)^{1/3}$.
7. $-x^{3/5}$.
8. $(-x)^{2/3}$.
9. $9x^{1/4}$.
10. $ab^{3/5}$.
11. $(-2a^2)^{1/3}$.
12. $-a^{1/3}b^{2/3}$.
13. $5^{1/2}(x+y)^{1/2}$.
14. $a(x-y)^{2/3}$.
15. $z(x-y)^{1/4}$.

Change each radical to an equivalent exponential form:

16. $\sqrt{a^3}$.
17. $\sqrt[3]{2a}$.
18. $\sqrt{3}$.
19. $\sqrt[3]{2a^2}$.
20. $\sqrt[4]{6a^3b}$.
21. $\sqrt{5xy^3}$.
22. $\sqrt{2(x+y)^5}$.
23. $\sqrt[5]{x-y}$.
24. $\sqrt[4]{x+y}$.
25. $\sqrt[3]{-a^2b^4}$.
26. $\sqrt{7xy}$.
27. $\sqrt[5]{-x^2y}$.
28. $\sqrt[3]{\sqrt{5}}$.
29. $\sqrt{3\sqrt{a}}$.
30. $\sqrt[3]{2\sqrt{3}}$.

Bring the coefficient, to the proper power, inside the radical sign:

31. $3\sqrt{2}$.
32. $2\sqrt[3]{3}$.
33. $2\sqrt[4]{x}$.
34. $2a\sqrt{3b}$.
35. $2\sqrt[n]{3}$.
36. $3xy\sqrt[3]{xy}$.
37. $\frac{x}{y}\sqrt{2xy^2}$.
38. $\frac{1}{3a}\sqrt{18a^3b}$.
39. $\frac{a^2}{b}\sqrt{\frac{3b^3}{a^3}}$.
40. $3x\sqrt{\frac{4-x}{9x^2}}$.
41. $2x\sqrt[3]{1-\frac{1}{x^3}}$.
42. $4a\sqrt{\frac{1}{16}-\frac{1}{a}}$.
43. $\frac{x+y}{2z}\sqrt{\frac{12z^3}{x+y}}$.
44. $\frac{a-b}{a+b}\sqrt{\frac{a+b}{a-b}}$.
45. $\frac{2}{a-b}\sqrt{\frac{(a-b)^3}{2}}$.

Express each radical in simplest form:

46. $\sqrt{8}$.
47. $\sqrt{20}$.
48. $\sqrt{50}$.
49. $\sqrt{4a^2b^5}$.
50. $\sqrt{27a^3b^2}$.
51. $\sqrt{40xy^2}$.
52. $\sqrt[3]{-16b}$.
53. $\sqrt[3]{81x^5y^6}$.
54. $\sqrt[3]{-64x^3y^2}$.
55. $\sqrt[4]{32b^5c^4}$.
56. $\sqrt[4]{81x^3y^5}$.
57. $\sqrt[4]{8a^5b^9}$.
58. $\sqrt{\frac{1}{3}}$.
59. $\sqrt{\frac{3}{2}}$.
60. $\sqrt{\frac{3}{8}}$.
61. $\sqrt[3]{\frac{1}{3}}$.
62. $\sqrt[3]{-\frac{3}{4}}$.
63. $\sqrt[3]{\frac{8}{9}}$.
64. $\sqrt{\frac{3}{5a^2}}$.
65. $\sqrt{\frac{2}{3x^2}}$.
66. $\sqrt{\frac{a^5x}{40y}}$.
67. $\sqrt[3]{\frac{7a^6b^4}{4c^2}}$.
68. $\sqrt[3]{\frac{18x^2y^2}{25z}}$.
69. $\sqrt[3]{\frac{-3a^4}{9b}}$.
70. $\sqrt{\frac{50x^5}{9yz^2}}$.
71. $\sqrt{\frac{5x^4}{6y^3z}}$.
72. $\sqrt{\frac{3c^4}{5ab^3}}$.

73. $\sqrt[4]{25}$. **74.** $\sqrt[4]{9a^2}$. **75.** $\sqrt[4]{49x^2}$.

76. $\sqrt[4]{4a^6b^8}$. **77.** $\sqrt[6]{8x^6y^9}$. **78.** $\sqrt[6]{81x^8y^6}$.

79. $\sqrt[6]{x^2y^4z^6}$. **80.** $\sqrt[6]{x^4y^6z^9}$. **81.** $\sqrt[6]{27a^3b^9}$.

82. $\sqrt[4]{\dfrac{4a^2}{9}}$. **83.** $\sqrt[4]{\dfrac{64a^6}{25}}$. **84.** $\sqrt[4]{\dfrac{16x^8}{y^2}}$.

Change each of the following expressions to one with a single radical sign:

85. $\sqrt[3]{\sqrt{a^3}}$. **86.** $\sqrt[4]{\sqrt[3]{x^4}}$. **87.** $\sqrt{\sqrt[3]{4a^2}}$.

88. $\sqrt[5]{\sqrt{81a^5b^{10}}}$. **89.** $\sqrt[3]{\sqrt[5]{27x^6y^6}}$. **90.** $\sqrt{\sqrt[4]{8a^6}}$.

61. Addition and Subtraction of Radicals. Radicals of the same order and the same radicand are called **like radicals.** The algebraic sum of like radicals is a radical of the same kind whose coefficient is the algebraic sum of the various coefficients. Unlike radicals often become like radicals when each is simplified. Then to make certain as to whether two or more radicals may be added, it is sufficient to express them in simplest form. Radicals not expressible as like radicals may have addition and subtraction indicated by the proper signs; they cannot be written as a single radical.

Example 1. Add algebraically $2\sqrt{18} - 3\sqrt{\frac{1}{2}} + \sqrt[4]{4}$.

Solution.

$$\begin{aligned} 2\sqrt{18} - 3\sqrt{\tfrac{1}{2}} + \sqrt[4]{4} &= 2\sqrt{9\cdot 2} - 3\sqrt{\tfrac{2}{4}} + \sqrt[4]{2^2} \\ &= 6\sqrt{2} - \tfrac{3}{2}\sqrt{2} + \sqrt{2} = \tfrac{11}{2}\sqrt{2}. \end{aligned}$$

Example 2. Add algebraically $\sqrt[3]{2a^4} - 3\sqrt[3]{16a} - \sqrt{2a}$.

Solution.

$$\begin{aligned} \sqrt[3]{2a^4} - 3\sqrt[3]{16a} - \sqrt{2a} &= a\sqrt[3]{2a} - 6\sqrt[3]{2a} - \sqrt{2a} \\ &= (a-6)\sqrt[3]{2a} - \sqrt{2a}. \end{aligned}$$

The unlike radicals $\sqrt[3]{2a}$ and $\sqrt{2a}$ cannot be combined into a single radical.

Exercise 38

Find the algebraic sum:

1. $4\sqrt{3} - \frac{3}{2}\sqrt{12} - \sqrt{27}$. **2.** $\sqrt{50} - \sqrt{18} + \sqrt{32}$.

3. $\sqrt{45} + \sqrt{80} + \sqrt{180}$. **4.** $3\sqrt{24} - \sqrt{150} - \sqrt{54}$.

5. $\sqrt{\frac{2}{3}} + 3\sqrt{6} - \sqrt{\frac{3}{2}}$. **6.** $\sqrt{\frac{1}{7}} + \sqrt{28} + \sqrt{112}$.

7. $3\sqrt{\frac{5}{3}} + 2\sqrt{\frac{3}{5}} + \sqrt{\frac{1}{15}}$. **8.** $5\sqrt{\frac{2}{5}} - 4\sqrt{\frac{5}{2}} + 5\sqrt{\frac{1}{10}}$.

9. $\sqrt{8x^3y^2} - 3\sqrt{18xy^2} + \sqrt{8x^3}$.

10. $\sqrt{4ab^3} - \sqrt{16a^3b} + \sqrt{ab}$.

11. $\sqrt[3]{54} + \sqrt[3]{16} - \sqrt[3]{250}$.

12. $\sqrt[3]{192} - \sqrt[3]{81} + \sqrt[3]{375}$.

13. $\sqrt[3]{-40} + \sqrt[3]{-135} + \sqrt[3]{-320}$.

14. $\sqrt[3]{-3a^4} - \sqrt[3]{-24a} - \sqrt[3]{-81a^4}$.

15. $\sqrt[3]{-8a} - \sqrt[3]{-27a^4} + \sqrt[3]{64a^7}$.

16. $\sqrt{a+b} + \sqrt{(a+b)^3}$.

17. $\sqrt{(x-y)^3} - \sqrt{x-y}$.

18. $\sqrt{\frac{1}{2}} - \sqrt{\frac{1}{4}} + \sqrt{\frac{1}{8}}$.

19. $\sqrt{\frac{1}{3}} + \sqrt[3]{\frac{1}{3}} + \sqrt{\frac{1}{27}}$.

20. $x\sqrt{\dfrac{x+y}{x-y}} - y\sqrt{\dfrac{x-y}{x+y}}$.

21. $\sqrt{\dfrac{x-y}{x+y}} + \sqrt{\dfrac{x+y}{x-y}}$.

22. $\sqrt{\dfrac{x-y}{2}} + \sqrt{\dfrac{2}{x-y}}$.

23. $\sqrt{6xy} + \sqrt{54xy^3} - \sqrt[3]{48x^4y}$.

24. $\sqrt[3]{3a^4} + 2\sqrt[3]{81a} - \sqrt{12a^3}$.

25. $\sqrt{108a^5} - \sqrt[3]{24a^4} - \sqrt{27a}$.

26. $\sqrt[6]{8x^3y^9} + \sqrt[4]{4x^2y^6} + \sqrt{2xy}$.

27. $\sqrt{3a^3b^3} - \sqrt[4]{9a^2b^6} - \sqrt[6]{27a^3b^9}$.

28. $\sqrt[3]{16x^4y} + \sqrt[6]{4x^2y^8} + \sqrt[9]{8x^3y^3}$.

62. Multiplication of Radicals. Since the $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$, the product of two or more radicals of the same order is a radical of the same order whose radicand is the product of the various radicands. To multiply radicals of different orders, it is first necessary to express them as radicals of the same order.

Example 1. Multiply $3\sqrt{2a}$ by $5\sqrt{3a^3}$.
Solution.

$$3\sqrt{2a}\cdot 5\sqrt{3a^3} = 15\sqrt{2a}\cdot\sqrt{3a^3} = 15\sqrt{6a^4} = 15a^2\sqrt{6}.$$

Example 2. Multiply $5\sqrt{3}$ by $6\sqrt[3]{2}$.
Solution. We take note of the different orders of the radicals and have

$$5\sqrt{3}\cdot 6\sqrt[3]{2} = (5\cdot 3^{1/2})(6\cdot 2^{1/3}) = 30\cdot 3^{3/6}\cdot 2^{2/6} = 30\sqrt[6]{3^3 2^2} = 30\sqrt[6]{108}.$$

To find this product we made use of fractional exponents. The result may be obtained, however, without resort to fractional exponents. To change the second order radical to an equivalent 6th radical, we multiply both the index of the radical and the exponent of the radicand by 3. Similarly, we use 2 as the multiplier in changing the third order radical to the 6th order. Thus we have

$$5\sqrt{3}\cdot 6\sqrt[3]{2} = 30\sqrt[6]{3^3}\sqrt[6]{2^2} = 30\sqrt[6]{3^3\cdot 2^2} = 30\sqrt[6]{108}.$$

Example 3. Find the product $(2\sqrt{3} + 3\sqrt{5}\,)(4\sqrt{3} - \sqrt{5}\,)$.
Solution. The binomials have like terms, and we multiply in the usual way.

$$\begin{aligned}(2\sqrt{3} + 3\sqrt{5}\,)(4\sqrt{3} - \sqrt{5}\,) &= 8\cdot 3 + 10\sqrt{15} - 3\cdot 5\\ &= 9 + 10\sqrt{15}.\end{aligned}$$

Exercise 39

Perform the indicated multiplications, leaving each result in simplest form:

1. $\sqrt{2}\cdot\sqrt{3}$.
2. $\sqrt{5}\cdot\sqrt{10}$.
3. $3\sqrt{30}\cdot\sqrt{3}$.
4. $3\sqrt{6x}\cdot\sqrt{2x}$.
5. $\sqrt{2xy}\cdot\sqrt{2y}$.
6. $\sqrt{7a}\cdot\sqrt{14ab}$.
7. $\frac{2}{5}\sqrt{15}\cdot\sqrt{5}$.
8. $\frac{2}{3}\sqrt{11}\cdot\frac{5}{2}\sqrt{11}$.
9. $6\sqrt{3}\cdot\sqrt{27}$.
10. $(-2\sqrt{5})^2$.
11. $(5\sqrt{2a})^2$.
12. $(3\sqrt{3x})^2$.
13. $(-2\sqrt[3]{3})^3$.
14. $(-2\sqrt[3]{-2})^3$.
15. $(-\sqrt[3]{4})^3$.
16. $\sqrt[3]{3a}\cdot\sqrt[3]{9a}$.
17. $\sqrt[3]{16a^2}\cdot\sqrt[3]{4a}$.
18. $\sqrt[3]{81}\cdot\sqrt[3]{9}$.
19. $\sqrt[3]{-4a}\cdot\sqrt[3]{4a^2}$.
20. $4\sqrt[3]{6}\cdot2\sqrt[3]{18}$.
21. $\frac{2}{3}\sqrt[3]{20}\cdot\sqrt[3]{4}$.
22. $(\sqrt[4]{4xy})^4$.
23. $\sqrt[3]{4xy^2}\cdot\sqrt[3]{2x^2y}$.
24. $\sqrt[4]{2xy}\cdot\sqrt[4]{4xy^3}$.
25. $\sqrt{2}\cdot\sqrt[3]{2}$.
26. $\sqrt{3}\cdot\sqrt[4]{3}$.
27. $\sqrt[3]{5}\cdot\sqrt{5}$.
28. $\sqrt[3]{3}\cdot\sqrt{\frac{1}{3}}$.
29. $\sqrt[3]{\frac{3}{5}}\cdot\sqrt{\frac{5}{3}}$.
30. $\sqrt{\frac{2}{3}}\cdot\sqrt[4]{6}$.
31. $\sqrt[3]{3}\cdot\sqrt[4]{3}$.
32. $\sqrt[4]{2}\cdot\sqrt{3}$.
33. $\sqrt[5]{5}\cdot\sqrt{5}$.
34. $(\sqrt{2}+\sqrt{3})^2$.
35. $(\sqrt{3}-\sqrt{4})^2$.
36. $(\sqrt{2}+\sqrt{5})^2$.
37. $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$.
38. $(\sqrt{4}+\sqrt{3})(\sqrt{4}-\sqrt{3})$.
39. $(2\sqrt{3}-4)(2\sqrt{3}+3)$.
40. $(4\sqrt{5}-5)(4\sqrt{5}+2)$.
41. $(5\sqrt{2}-\sqrt{3})(5\sqrt{2}+2\sqrt{3})$.
42. $(2\sqrt{6}+\sqrt{5})(\sqrt{6}-4\sqrt{5})$.
43. $(2\sqrt{6}-3\sqrt{2})(3\sqrt{6}+\sqrt{2})$.
44. $(\sqrt{10}-3\sqrt{2})(2\sqrt{10}+4\sqrt{2})$.
45. $(\sqrt{2}+\sqrt{3}-\sqrt{5})^2$.
46. $(\sqrt{2}-\sqrt{5}+\sqrt{7})^2$.

63. Division of Radicals. The relation

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

shows how to express the quotient of two radicals of the same order as a single radical. But, whether we consider the left member of the above equation or the right member, it is customary to rationalize the denominator. That is, an equivalent form is to be obtained which has no radical in the denominator, and no radicand is to be a fraction. It turns out, therefore, that the process of division where radicals are involved as divisors is largely that of rationalizing the denominators. To divide radicals of different orders it is first necessary to express them as radicals of the same order. We shall illustrate with some examples.

Example 1. Divide $5\sqrt{2}$ by $3\sqrt{6}$.

Solution.

$$\frac{5\sqrt{2}}{3\sqrt{6}}=\frac{5\sqrt{2}}{3\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}}=\frac{5\sqrt{12}}{3\sqrt{36}}=\frac{10\sqrt{3}}{18}=\frac{5}{9}\sqrt{3}.$$

Example 2. Divide $\sqrt[4]{2}$ by $\sqrt[3]{3^2}$.

Solution. The radicals must first be changed to radicals of the same order. The L.C.M. of 3 and 4 is 12; hence we express each of the given radicals as an equivalent radical of the 12th order:

$$\frac{\sqrt[4]{2}}{\sqrt[3]{3^2}} = \frac{2^{1/4}}{3^{2/3}} = \frac{2^{3/12}}{3^{8/12}} = \frac{\sqrt[12]{2^3}}{\sqrt[12]{3^8}} = \sqrt{\frac{2^3}{3^8}} = \sqrt[12]{\frac{2^3 \cdot 3^4}{3^8 \cdot 3^4}} = \frac{\sqrt[12]{648}}{3}.$$

When the divisor is a binomial which contains a radical in one or both terms we *rationalize the divisor* by multiplying both the dividend and divisor by a properly chosen expression. This process is also known as *rationalizing the denominator* of the fraction formed by the dividend and divisor. For this purpose we recall that the product of $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ is the rational expression $a - b$. Hence a rationalizing factor of a divisor of the kind in question is obtained by changing the sign of one term of the divisor.

Example 3. Divide $(2\sqrt{3} - 3\sqrt{2})$ by $(3\sqrt{3} - 4\sqrt{2})$.

Solution. The expression $(3\sqrt{3} + 4\sqrt{2})$ is a rationalizing factor of the divisor. Writing the dividend and divisor as the numerator and denominator of a fraction, we obtain

$$\frac{2\sqrt{3} - 3\sqrt{2}}{3\sqrt{3} - 4\sqrt{2}} = \frac{2\sqrt{3} - 3\sqrt{2}}{3\sqrt{3} - 4\sqrt{2}} \cdot \frac{3\sqrt{3} + 4\sqrt{2}}{3\sqrt{3} + 4\sqrt{2}} = \frac{18 - \sqrt{6} - 24}{27 - 32}$$

$$= \frac{-\sqrt{6} - 6}{-5} = \frac{\sqrt{6} + 6}{5}.$$

The final fraction, containing no radical in the denominator, gives the quotient in the desired form.

Exercise 40

Perform the indicated divisions, giving each result in simplest form:

1. $\sqrt{6} \div \sqrt{2}$.
2. $\sqrt{14} \div \sqrt{21}$.
3. $\sqrt{10} \div \sqrt{5}$.
4. $\sqrt{26} \div \sqrt{2}$.
5. $\sqrt{5x} \div \sqrt{3x}$.
6. $\sqrt[3]{4x^2y} \div \sqrt[3]{2xy}$.
7. $\sqrt[3]{7xy^2} \div \sqrt[3]{4x^2y}$.
8. $9 \div 3\sqrt{5}$.
9. $3 \div \sqrt[3]{5}$.
10. $4\sqrt[3]{5} \div 5\sqrt[3]{9}$.
11. $\sqrt[4]{2xy} \div \sqrt[4]{x^2y}$.
12. $\sqrt[4]{6} \div \sqrt[4]{8}$.
13. $\frac{5}{6}\sqrt{\frac{3}{14}} \div \frac{2}{3}\sqrt{\frac{2}{7}}$.
14. $\frac{1}{2}\sqrt[4]{\frac{1}{2}} \div \frac{1}{3}\sqrt[4]{\frac{1}{3}}$.
15. $\frac{2}{3}\sqrt[3]{\frac{3}{2}} \div \frac{4}{5}\sqrt[3]{\frac{5}{4}}$.
16. $\sqrt{3} \div \sqrt[3]{3}$.
17. $\sqrt{7} \div \sqrt[3]{14}$.
18. $\sqrt[3]{25} \div \sqrt{5}$.
19. $6\sqrt[4]{9} \div 3\sqrt{3}$.
20. $\sqrt[3]{6} \div \sqrt[4]{2}$.
21. $4\sqrt{22} \div \sqrt[4]{11}$.

22. $(2\sqrt{3} + \sqrt{2}) \div \sqrt{6}$.
23. $(2\sqrt{30} - 3\sqrt{70}) \div \sqrt{10}$.
24. $(7\sqrt{13} - 2\sqrt{26}) \div 3\sqrt{13}$.
25. $(9\sqrt{14} + 6\sqrt{21}) \div 3\sqrt{42}$.
26. $(\frac{3}{2}\sqrt{3} + \frac{2}{3}\sqrt{2}) \div \frac{1}{6}\sqrt{6}$.
27. $(\frac{1}{2}\sqrt{3} - \frac{2}{3}\sqrt{5}) \div \sqrt{15}$.
28. $1 \div (1 - \sqrt{3})$.
29. $1 \div (2 + \sqrt{2})$.
30. $3 \div (3 - \sqrt{2})$.
31. $4 \div (\sqrt{5} - \sqrt{3})$.
32. $\sqrt{2} \div (3 - \sqrt{2})$.
33. $\sqrt{7} \div (\sqrt{5} - 1)$.
34. $2\sqrt{3} \div (3\sqrt{2} + \sqrt{3})$.
35. $6\sqrt{2} \div (2\sqrt{3} + 3\sqrt{2})$.

Rationalize the denominator of each fraction. Then use Table I, page 301, to find the values of fractions **36–42** to three decimal places.

36. $\dfrac{\sqrt{5} + 1}{\sqrt{5} - 1}$.
37. $\dfrac{5 + \sqrt{2}}{2 - \sqrt{2}}$.
38. $\dfrac{\sqrt{3} - 4}{\sqrt{2} + 4}$.
39. $\dfrac{\sqrt{6} - 2\sqrt{3}}{\sqrt{2} - \sqrt{3}}$.
40. $\dfrac{3\sqrt{3} - 2\sqrt{2}}{2\sqrt{3} - 4\sqrt{2}}$.
41. $\dfrac{2\sqrt{7} + 3\sqrt{5}}{\sqrt{7} + \sqrt{5}}$.
42. $\dfrac{3\sqrt{2} + 4\sqrt{6}}{2\sqrt{5} + 3\sqrt{6}}$.
43. $\dfrac{\sqrt{2} + \sqrt{x}}{\sqrt{2} - \sqrt{x}}$.
44. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$.
45. $\dfrac{3\sqrt{x} + 2\sqrt{y}}{2\sqrt{x} - 3\sqrt{y}}$.
46. $\dfrac{2 - \sqrt{6}}{(3 - \sqrt{6})(2 + \sqrt{6})}$.
47. $\dfrac{\sqrt{3} + 2}{(2 - \sqrt{3})(1 + \sqrt{3})}$.
48. $\dfrac{1}{1 + \sqrt{2} + \sqrt{3}}$.
49. $\dfrac{2}{\sqrt{2} + \sqrt{3} - \sqrt{5}}$.
50. $\dfrac{\sqrt{5} - \sqrt{2} + \sqrt{3}}{\sqrt{5} + \sqrt{2} + \sqrt{3}}$.

Hint: First multiply the numerator and denominator of problem **48** by the expression $1 + \sqrt{2} - \sqrt{3}$.

64. Imaginary Numbers. In the treatment of fractional exponents and radicals we avoided the even roots of negative numbers. But even roots, especially square roots, of negative numbers have important uses. The solution of certain equations involves this kind of number. For example, solving the equation $x^2 + 1 = 0$, we have

$$x^2 = -1,$$
$$x = \pm\sqrt{-1}.$$

We now consider radicals of order two where the radicands are negative. That is, we introduce numbers like

$$\sqrt{-1}, \quad \sqrt{-3}, \quad \sqrt{-b}, \quad \text{where} \quad b > 0.$$

Each of these radicals stands for the square root of a negative number. Consequently their squares must yield the negative numbers. Thus,

$$(\sqrt{-1})^2 = -1, \quad (\sqrt{-3})^2 = -3, \quad (\sqrt{-b})^2 = -b$$

But the square of a real number cannot be negative. Hence a number indicated by $\sqrt{-b}$, $b > 0$, is not real. It is called an **imaginary number.**

Frequently the letter i is used to stand for the number $\sqrt{-1}$. Employing i in this way, we shall assume that the following operations apply:

$$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\sqrt{-1} = 3i,$$

$$\sqrt{-2} = \sqrt{(-1)(2)} = \sqrt{-1}\sqrt{2} = i\sqrt{2},$$

$$\sqrt{-b} = \sqrt{(-1)(b)} = \sqrt{-1}\sqrt{b} = i\sqrt{b} \quad (b > 0).$$

Thus we see that the square root of a negative number can be expressed as a real number times i. The number i, or $\sqrt{-1}$, is called the **imaginary unit.** By definition the square of this unit is -1.

Since $i^2 = -1$, we may easily find the value of any integral power of i. We have

$$\begin{aligned} i^1 &= i, & i^5 &= i^4 \cdot i = i,\\ i^2 &= -1, & i^6 &= i^4 \cdot i^2 = (1)(-1) = -1,\\ i^3 &= i^2 \cdot i = (-1)i = -i, & i^7 &= i^6 \cdot i = (-1)i = -i,\\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1, & i^8 &= i^4 \cdot i^4 = (1)(1) = 1. \end{aligned}$$

An examination of these powers of i shows that only the numbers i, -1, $-i$, and 1 occur in the successive positive integral powers of i; and that i with any exponent remains unchanged in value when the exponent is increased, or decreased, by 4.

A number of the form $a + bi$, where a and b are real numbers, is called a **complex number.** The number a is called the **real part** and bi is called the **imaginary part.** For the special case in which $b = 0$, the complex number is real; and for $b \neq 0$, the complex number is called imaginary. If $a = 0$ and $b \neq 0$, the number is called a **pure imaginary.** Thus 3, $2 - 5i$, and $6i$ are complex numbers; 3 is a real number, $2 - 5i$ is an imaginary number, and $6i$ is a pure imaginary number. Complex numbers, we note, comprise both the real numbers and the imaginary numbers.

The numbers $a + bi$ and $a - bi$ are said to be **conjugate** complex numbers. Each is the conjugate of the other. We shall encounter conjugate complex numbers later in solving quadratic equations.

Two complex numbers are defined to be equal if and only if their real parts are equal and the imaginary parts are equal. That is,

$$a + bi = c + di$$

if and only if

$$a = c \quad \text{and} \quad b = d.$$

From this definition it follows that if

$$a + bi = 0,$$

then

$$a = 0 \quad \text{and} \quad b = 0.$$

Conversely, if $a = b = 0$, then $a + bi = 0$. Hence a complex number is equal to zero if and only if the real and imaginary parts are both zero.

The usual rules for addition, subtraction, multiplication, and division are applied to complex numbers with imaginary parts. To perform such operations, it is preferable first to express the numbers in the form $a + bi$. Then the quantity i is handled just as any literal number with the additional rule that -1 is to replace i^2 wherever it arises.

Example 1. Add the numbers $2 - 3\sqrt{-3}$ and $-4 + \sqrt{-3}$.

Solution. We first express each number in the form $a + bi$. Thus we obtain

$$\begin{aligned}(2 - 3\sqrt{-3}\,) + (-4 + \sqrt{-3}\,) &= (2 - 3i\sqrt{3}\,) + (-4 + i\sqrt{3}\,)\\ &= -2 - 2i\sqrt{3}.\end{aligned}$$

Example 2. Multiply $\sqrt{-3}$ by $\sqrt{-5}$.

Solution.

$$\sqrt{-3}\cdot\sqrt{-5} = i\sqrt{3}\cdot i\sqrt{5} = i^2\sqrt{15} = -\sqrt{15}.$$

NOTE. The law for multiplying two radicals of like order involving real numbers must not be applied to two imaginary numbers. Thus $\sqrt{3}\cdot\sqrt{5} = \sqrt{15}$, but $\sqrt{-3}\cdot\sqrt{-5}$ is not equal to $\sqrt{15}$. An error in a product is less likely to be made if each imaginary part is expressed as a real number times i before multiplying.

Example 3. Multiply $3 - 4i$ by $5 + 5i$.

Solution.

$$\begin{aligned}(3 - 4i)(5 + 5i) &= 15 - 5i - 20i^2\\ &= 15 - 5i - 20(-1)\\ &= 35 - 5i.\end{aligned}$$

The product of two conjugate complex numbers is a real number. Thus

$$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2.$$

We shall make use of this property of conjugates to find the quotient of two numbers when the divisor is imaginary.

Example 4. Divide $2 + 3i$ by $4 - 5i$.

Solution. For the quotient to be in the form $a + bi$, it is necessary that the denominator have no imaginary quantity. The product of the denominator $4 - 3i$ and its conjugate $4 + 3i$ is real. Hence we write

$$\begin{aligned}\frac{2+3i}{4-5i} &= \frac{(2+3i)(4+5i)}{(4-5i)(4+5i)}\\ &= \frac{8+22i+15i^2}{16-25i^2}\\ &= \frac{-7+22i}{16+25} = -\frac{7}{41}+\frac{22}{41}i.\end{aligned}$$

Exercise 41

1. Classify each of the following complex numbers as a real, an imaginary, or a pure imaginary number: -5, $\sqrt{2}$, $2 - i$, $2 + \sqrt{-5}$, $3i$, π, $\sqrt[3]{-4}$, $\sqrt{-2}$.

Write each of the following numbers in the form $a + bi$:

2. $3 + \sqrt{-9}$.
3. $4 - \sqrt{-36}$.
4. $5 + \sqrt{-25}$.
5. $\sqrt{-100}$.
6. $-7\sqrt{-16}$.
7. $10\sqrt{-81}$.
8. $\sqrt{-18}$.
9. $-\sqrt{-45}$.
10. $-\sqrt{-x^2y^4}$.
11. $2 - 3\sqrt{-5}$.
12. $\sqrt{-\frac{1}{2}}$.
13. $3 + 4\sqrt{-27}$.

Perform the indicated operations, leaving the results in the form $a + bi$:

14. $3\sqrt{-4} + \sqrt{-36} - \sqrt{-9}$.
15. $3\sqrt{-1} - 4\sqrt{-4} + 9\sqrt{-9}$.
16. $2\sqrt{-8} - 3\sqrt{-32} + 5\sqrt{-2}$.
17. $\sqrt{-27} - 4\sqrt{-12} + 6\sqrt{-3}$.
18. $(3 + 4i) + (-7 - 5i)$.
19. $(-4 + 5i) - (-6 + 8i)$.
20. $(9 - 4i) - (-8 + 6i)$.
21. $(10 + 7i) + (11 + 10i)$.

22. $(6 - 2\sqrt{-1}) + (3 - 3\sqrt{-16}) - (7 - 5\sqrt{-9})$.
23. $(1 - 3\sqrt{-64}) - (-4 + 5\sqrt{-81}) + (12 + 9\sqrt{-4})$.
24. $(2 - \sqrt{-5}) + (8 + 2\sqrt{-20}) - (11 - 3\sqrt{-45})$.

25. $(2 + 3i)(3 + 2i)$.
26. $(4 + 3i)(-1 + i)$.
27. $(3 - i)(2 - i)$.
28. $(4 - 3i)(4 - 3i)$.
29. $(x - yi)^2$.
30. $(1 - 3i)(1 + 3i)$.
31. $i^4 + i^5 - i^6$.
32. $i^{11} + i^{12} + i^{13}$.
33. $i^{17} \div i^{14}$.
34. $3 \div (1 + 2i)$.
35. $(2 + i) \div i$.
36. $5 \div (2 - 3i)$.
37. $(1 + i) \div (1 - i)$.
38. $(3 - 2i) \div (2 + i)$.
39. $(7 + 2i)(2 - 7i)$.

40. $(2 + i\sqrt{3})(3 - 2i\sqrt{3})$.
41. $(\sqrt{2} + 3i)(\sqrt{2} - 4i)$.
42. $(5 - i\sqrt{2})(5 - i\sqrt{2})$.
43. $(10 - i\sqrt{3}) \div (1 + i\sqrt{3})$.
44. $(3 - 5i\sqrt{2}) \div (2 - 3i\sqrt{2})$.
45. $\sqrt{-2}\cdot\sqrt{-8}$.
46. $(-3\sqrt{-4})(-2\sqrt{-9})$.
47. $\sqrt{-3}\cdot\sqrt{-4}\cdot\sqrt{-5}$.

Find the sum, the difference (first number minus the second), and the product of each pair of conjugate complex numbers:

48. $2 - i,\ 2 + i$. **49.** $5 - 2i,\ 5 + 2i$. **50.** $a + bi,\ a - bi$.

51. Show by substituting for x that $2 + 3i$ and $2 - 3i$ are roots of the equation $x^2 - 4x + 13 = 0$.

52. Show that $1 + 4i$ and $1 - 4i$ are roots of $x^2 - 2x + 17 = 0$.

Supplementary Problems

Simplify by performing the indicated operations and leaving the results without zero or negative exponents:

1. $a^3(-ab^2)^2(-a^2b^3)^3$. **2.** $3x^3(-x)^2(2x^2)^2$. **3.** $(2x)^3(-3x^2)^2(-x)^3$.
4. $x^{n+4} \cdot x^{-n+3}$. **5.** $a^{m-6} \cdot a^{m+7}$. **6.** $a^{3n-2} \cdot a^{3-2n}$.
7. $4^{1-2n} \cdot 4^{2+n}$. **8.** $(3^{n+2})^2$. **9.** $(2a)^m(2a^n)$.

10. $(x^{-3} + y^{-3})(x^{-3} - y^{-3})$. **11.** $(2^{-2} + 3^{-2})(2^{-2} - 3^{-2})$.

12. $(x^{-1} - y^{-1})(x^{-1} - y^{-1})$. **13.** $\dfrac{18x^3y^4z^2}{6xyz^2}$.

14. $\dfrac{32a^2b^3c^4}{8a^0b^2c^3}$. **15.** $\dfrac{21a^5b^4c^4}{7a^4bc^3}$. **16.** $\dfrac{5^3x^4y^3}{(5xy^2)^2}$.

17. $\dfrac{(3x^3y^4)^3}{(9x^2y^3)^2}$. **18.** $\dfrac{(9a^3b^6)^3}{(3a^3b^5)^3}$. **19.** $\left(\dfrac{7x^3y^4}{5z^3}\right)\left(\dfrac{25y^4z}{14x^4}\right)$.

20. $\left(\dfrac{3x^4y^3}{5yz^5}\right)\left(\dfrac{10yz}{9x^2y}\right)$. **21.** $\left(\dfrac{6x^4y^2}{7xz^2}\right)\left(\dfrac{35z^6}{3x^3y^4}\right)$. **22.** $\left(\dfrac{2r^3s^2}{3t^4}\right)^3\left(\dfrac{9t^7}{2r^2s^3}\right)^2$.

23. $\left(\dfrac{2^0r^2t^3}{3r^0t^4}\right)^2\left(\dfrac{r^4t^5}{2r^5t^4}\right)^3$. **24.** $\left(\dfrac{2^3r^3s^3}{4^2r^0s}\right)\left(\dfrac{x}{y}\right)^0$. **25.** $\dfrac{3^{-2}a^3b^{-3}}{2^{-3}a^{-1}b^2}$.

26. $\dfrac{(-2)^3(-p)^3q^{-1}}{(-3)^2p^{-2}q^{-3}}$. **27.** $\dfrac{4^{-1}x^3y^{-2}}{3^{-2}x^0y^{-4}}$. **28.** $\left(\dfrac{3xy}{2x^{-2}y^3}\right)^{-2}$.

29. $\left(\dfrac{m^{-3}n^{-2}}{x^0y^{-1}}\right)^{-3}$. **30.** $\left(\dfrac{2^{-2}a^{-2}c^{-3}}{3^{-2}a^{-3}b^{-1}}\right)^2$. **31.** $\dfrac{(a^2b^{-3}c^{-1})^4}{(ab^{-1}c^3)^3}$.

32. $\dfrac{(2a^{-1}b^3)^{-2}}{(3ab^{-1})^{-3}}$. **33.** $\dfrac{(3a^{-4}b^6)^{-3}}{(6^{-1}a^3b^{-4})^3}$. **34.** $\dfrac{1^{-1} + 2^{-1}}{3^{-1}}$.

35. $\dfrac{2^{-1} - 2^{-3}}{2^{-4}}$. **36.** $\dfrac{3^{-1} - 2^{-1}}{3^{-2} + 2^{-2}}$. **37.** $\dfrac{x^{-1} - y^{-1}}{(x - y)^{-1}}$.

38. $\dfrac{a^{-3} + b^{-3}}{a^{-2} + b^{-2}}$. **39.** $\dfrac{x^{-1} - y^{-1}}{x^{-2} - y^{-2}}$. **40.** $a^{2/3}a^{1/3}a^{-1}$.

41. $a^{2/3}(-a)^2a^{-5/2}$. **42.** $c^{3/8}c^{-5/8}c$. **43.** $(x^{-1/2}y^{-3/4})^4$.
44. $(2^{-1/2}x^{2/3}y^{5/6})^{-6}$. **45.** $(3^{1/3}x^{-1/6}y^{7/2})^{12}$. **46.** $(81x^4y^2)^{1/2}$.
47. $(64x^{-6}y^3)^{1/3}$. **48.** $(81x^{-8}y^{12})^{1/4}$. **49.** $(5^{-1/2}a^{-2/3}b^{3/2})^6$.

50. $(2^{-1}a^{3/4}b^{-1/4})^4$. **51.** $(4^{1/6}a^{-5/6}b^{1/3})^3$. **52.** $\dfrac{8^{1/3}a^{2/3}}{4^{-1/2}a^{-1/3}}$.

53. $\dfrac{(-27)^{-2/3}b^{-3}}{b^{-2/3}}$. **54.** $\dfrac{8a^{-4}b^{2/3}}{4^{-3/2}a^{-2}b^{-5/6}}$. **55.** $\left(\dfrac{2x^{-3/8}}{3x^{-1/4}}\right)^{-4}$.

56. $\left(\dfrac{x^0y^{-3/5}}{x^2y^{-4/5}}\right)^{-10}$. **57.** $\left(\dfrac{2^{-1/3}x^{1/2}y^{3/2}}{3^{1/3}x^{-3/2}y^{1/2}}\right)^6$. **58.** $\left(\dfrac{4x^{-1}y^{-1/2}}{16^{-1}x^{-3}y^{5/2}}\right)^{1/2}$.

59. $\left(\frac{9x^{-3/8}y^{3/4}}{3^3x^{1/4}y^{-3/4}}\right)^{-1/2}$.

60. $\left(\frac{4x^{-3/2}y}{2^{-1}x^{-1/2}y^{-1/3}}\right)^{-2/3}$.

61. $(x^{1/3} + y^{1/2})(x^{-1/2} + y^{-1/3})$.

62. $(x^{-3/2} - y^{3/2})(x^{3/2} + y^{-3/2})$.

63. $(x^{2/3} - 1)(x^{2/3} + 3)$.

64. $(2x^{1/2} + 3)(3x^{-1/2} - 2)$.

Express each of the following in simplest form:

65. $\sqrt{\frac{5a^3}{8b^2}}$.

66. $\sqrt{\frac{32x^3}{27y^3}}$.

67. $\sqrt[3]{\frac{-16x^4}{9y}}$.

68. $\sqrt[4]{\frac{25a^7}{2b^8}}$.

69. $\sqrt[4]{\frac{8a^3}{x^7}}$.

70. $\sqrt[3]{\frac{-8}{27x^2y^3}}$.

71. $\sqrt[5]{\frac{-2x^4}{ay^{11}}}$.

72. $\sqrt[7]{\frac{-3a^7}{64b^{11}}}$.

73. $\sqrt{\frac{a^2 + b^2}{a^2 - b^2}}$.

74. $\sqrt[4]{4x^{10}y^4}$.

75. $\sqrt[n]{2^nx^{2n}y^{3n}}$.

76. $\sqrt[2n]{x^{2n}y^{6n}}$.

Perform the indicated operations, leaving each result in simplest form:

77. $2\sqrt{50} - 3\sqrt{18} + 4\sqrt{2}$.

78. $\sqrt{48} + \sqrt{12} - 3\sqrt{27}$.

79. $5\sqrt{24} - \sqrt{150} - \sqrt{96}$.

80. $\sqrt{252} - 2\sqrt{175} + \sqrt{112}$.

81. $\sqrt[3]{16} + \sqrt[3]{24} + \sqrt[3]{40}$.

82. $\sqrt[3]{-32} + \sqrt[3]{108} + 2\sqrt[3]{4}$.

83. $2\sqrt{\frac{1}{27}} + \sqrt{\frac{1}{12}} - 3\sqrt{\frac{1}{3}}$.

84. $\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{6}} - \sqrt{\frac{3}{2}}$.

85. $\sqrt{2xy^3} + \sqrt{8xy^3} + y\sqrt{18xy}$.

86. $x^2\sqrt{12x} - x\sqrt{27x^3} - \sqrt{48x^5}$.

87. $\sqrt{\frac{2}{x}} + \sqrt{\frac{x}{2}} + \sqrt{\frac{1}{2x}}$.

88. $b\sqrt{\frac{8a}{b}} - a\sqrt{\frac{b}{2a}} + ab\sqrt{\frac{2}{ab}}$.

89. $\sqrt{\frac{x^3}{2y}} + \sqrt{\frac{x}{2y^3}} + \sqrt{\frac{2x^3}{y^5}}$.

90. $x\sqrt{\frac{x - y}{x + y}} + y\sqrt{\frac{x + y}{x - y}}$.

Chapter Nine

LOGARITHMS

65. Logarithms. The laws of exponents form the basis of the study of logarithms, whose study, in turn, will enable us to solve with ease many complicated numerical problems. Further, we shall learn how to extract roots of any rational order with a simplicity that will surprise the student who remembers his previous experience with roots of the second order.

DEFINITION OF A LOGARITHM. If b is a positive number other than 1 and x is a real number in the equation $b^x = N$, then x is called the **logarithm** of N to the base b. This relation may be written symbolically as $x = \log_b N$. Thus the two equations

$$b^x = N \quad \text{and} \quad x = \log_b N$$

express the same relation among the numbers b, x, and N. The first is in **exponential form** and the second is in **logarithmic form.**

In this definition it is specified that $b \neq 1$ and that the exponent x is a real number. The reason for rejecting 1 as a base is that any power of 1 is still 1. Concerning the exponent, we recall that real numbers include both the rational and irrational numbers. We have not defined an irrational exponent. In view of this fact, we make the following statements which are proved in more advanced mathematics.

1. Irrational exponents can be defined so that they obey the five laws of exponents (Section 56).
2. If x is an irrational number, an approximation for b^x is obtained by using a rational approximation for x. An approximate value for b^x can be found to any desired degree of accuracy by choosing a sufficiently close rational approximation to the irrational exponent x. For example, $3^{\sqrt{2}}$ and $3^{1.414}$ are nearly equal. A better approximation is obtained when more places are used in the decimal approximation of $\sqrt{2}$.

These assumptions enable us to conclude that b^x, where $b > 0$ and x is real, is a positive quantity. We remember, of course, that the principal root is to be selected in case of a fractional exponent. Hence we shall consider logarithms of positive numbers only. It is also true, though the proof is omitted, that the equation $b^x = N$, where N and b are positive and $b \neq 1$, is satisfied by one and only one real value of x.

As illustrations of the meaning of logarithms, we list some exponential relations and the corresponding equivalent logarithmic relations.

Exponential Form	*Logarithmic Form*
$2^3 = 8$	$\log_2 8 = 3$
$8^{2/3} = 4$	$\log_8 4 = \frac{2}{3}$
$5^{-2} = \frac{1}{25}$	$\log_5 \frac{1}{25} = -2$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$

Example 1. Find N if $\log_3 N = -4$.

Solution. Using the equivalent exponential form, we have

$$N = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}.$$

Example 2. Find b if $\log_b 25 = \frac{2}{3}$.

Solution. The exponential form gives $b^{2/3} = 25$. Now the $\frac{3}{2}$ power of both numbers gives $(b^{2/3})^{3/2} = 25^{3/2}$, and $b = 125$.

Example 3. Find the value of $\log_7 \frac{1}{49}$.

Solution. Let $x = \log_7 \frac{1}{49}$. Then $7^x = \frac{1}{49} = 7^{-2}$, and $x = -2$.

Exercise 42

Give the logarithmic form of the following:

1. $2^4 = 16$.
2. $5^2 = 25$.
3. $3^3 = 27$.
4. $2^{-3} = \frac{1}{8}$.
5. $6^{-1} = \frac{1}{6}$.
6. $10^{-2} = \frac{1}{100}$.
7. $10^1 = 10$.
8. $3^1 = 3$.
9. $8^0 = 1$.
10. $36^{1/2} = 6$.
11. $64^{1/3} = 4$.
12. $16^{5/4} = 32$.
13. $9^{-1/2} = \frac{1}{3}$.
14. $8^{-2/3} = \frac{1}{4}$.
15. $4^{-3/2} = \frac{1}{8}$.
16. $(\frac{2}{3})^{-2} = \frac{9}{4}$.
17. $(\frac{3}{2})^2 = \frac{9}{4}$.
18. $(\frac{1}{25})^{-3/2} = 125$.

Give the exponential form of the following:

19. $\log_2 4 = 2$.
20. $\log_6 36 = 2$.
21. $\log_5 125 = 3$.
22. $\log_9 3 = \frac{1}{2}$.
23. $\log_8 32 = \frac{5}{3}$.
24. $\log_{25} 125 = \frac{3}{2}$.
25. $\log_{10} 10 = 1$.
26. $\log_4 1 = 0$.
27. $\log_5 1 = 0$.
28. $\log_3 \frac{1}{9} = -2$.
29. $\log_2 \frac{1}{8} = -3$.
30. $\log_5 \frac{1}{625} = -4$.
31. $\log_{1/3} 27 = -3$.
32. $\log_{1/2} 32 = -5$.
33. $\log_{1/6} 36 = -2$.

Give the value of each of the following:

34. $\log_2 16$.
35. $\log_5 25$.
36. $\log_6 216$.
37. $\log_{25} 5$.
38. $\log_{125} 5$.
39. $\log_{36} 6$.
40. $\log_{10} 10$.
41. $\log_6 6$.
42. $\log_{64} 2$.
43. $\log_9 1$.
44. $\log_7 \frac{1}{49}$.
45. $\log_{10} \frac{1}{100}$.

Find b or N in each of the following:

46. $\log_{10} N = 2$.
47. $\log_5 N = 1$.
48. $\log_3 N = 4$.
49. $\log_4 N = \frac{1}{2}$.
50. $\log_{10} N = 1$.
51. $\log_9 N = \frac{3}{2}$.
52. $\log_{1/2} N = 4$.
53. $\log_{1/3} N = -2$.
54. $\log_5 N = -2$.
55. $\log_{10} N = 0$.
56. $\log_{1/4} N = -3$.
57. $\log_9 N = -\frac{3}{2}$.
58. $\log_b 25 = 2$.
59. $\log_b 4 = 1$.
60. $\log_b 8 = 3$.
61. $\log_b 9 = \frac{1}{2}$.
62. $\log_b 4 = -2$.
63. $\log_b 25 = -1$.
64. $\log_b 9 = -\frac{2}{3}$.
65. $\log_b 16 = -\frac{4}{3}$.
66. $\log_b 8 = -\frac{3}{5}$.

66. Laws of Logarithms. We shall state and prove three laws of logarithms which are especially useful in numerical computations. The laws follow readily from the laws of exponents.

LAW I. *The logarithm of a product is equal to the sum of the logarithms of the factors.*

Proof: Suppose that M and N are the factors of the product and that

$$\log_b M = x \quad \text{and} \quad \log_b N = y, \quad \text{or} \quad M = b^x \quad \text{and} \quad N = b^y.$$

Then

$$MN = b^x \cdot b^y = b^{x+y} \quad \text{and} \quad \log_b MN = x + y.$$

Substituting for x and y, we obtain

$$\boldsymbol{\log_b MN = \log_b M + \log_b N.}$$

For three factors, as MNP, we have

$$\begin{aligned} \log_b (MN)P &= \log_b MN + \log_b P \\ &= \log_b M + \log_b N + \log_b P. \end{aligned}$$

Similarly, the law may be extended to any number of factors.

Illustration: $\log_{10} (856)(72.2) = \log_{10} 856 + \log_{10} 72.2$.

LAW II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

Proof: Let $\log_b M = x$ and $\log_b N = y$, or $M = b^x$ and $N = b^y$. Hence

$$\frac{M}{N} = \frac{b^x}{b^y} = b^{x-y} \quad \text{and} \quad \log_b \frac{M}{N} = x - y.$$

By substituting for x and y, we obtain

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

Illustration: $\log_2 \frac{69.3}{708} = \log_2 69.3 - \log_2 708.$

Law III. *The logarithm of a power of a number is equal to the exponent times the logarithm of the number.*

Proof: Let $\log_b M = x$, or $M = b^x$. Then

$$M^p = (b^x)^p = b^{px},$$

and

$$\log_b M^p = px.$$

Substituting for x gives

$$\log_b M^p = p \log_b M.$$

Illustrations:

$$\log_{10} 87^3 = 3 \log_{10} 87$$

$$\log_3 59^{-5} = -5 \log_3 59$$

$$\log_{10} \sqrt{459} = \log_{10} 459^{1/2} = \tfrac{1}{2} \log_{10} 459$$

$$\log_5 \frac{(28.5)^2(7.63)^{1/4}}{(58.7)^{2/3}} = 2 \log_5 28.5 + \frac{1}{4} \log_5 7.63 - \frac{2}{3} \log_5 58.7$$

Exercise 43

Write each of the following expressions as the sum or difference of logarithms. A base is not indicated. Any positive number except 1 may be used.

1. $\log (852)(561)(473)$.

2. $\log (78)^3(7.12)$.

3. $\log \frac{(85)(78)}{69^{2/3}}$.

4. $\log \frac{83}{(51)(47)}$.

5. $\log \frac{(14)(33)}{(28)(61)}$.

6. $\log \frac{(8.23)^2(1.44)^{1/4}}{308}$.

7. $\log \frac{\sqrt{57} \cdot \sqrt[3]{91}}{(18.4)^4}$.

8. $\log \sqrt{\frac{(28)(44)(21)}{978}}$.

9. $\log \sqrt{\frac{(84)(61)}{37}}$.

10. $\log \frac{85\sqrt[3]{67}}{43^2\sqrt{65}}$.

Given $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find the value of each of the following:

11. $\log_{10} 8$.

12. $\log_{10} 9$.

13. $\log_{10} 6$.

14. $\log_{10} 48$.

15. $\log_{10} 54$.

16. $\log_{10} \sqrt[3]{12}$.

17. $\log_{10} \frac{16}{3}$.

18. $\log_{10} \frac{27}{8}$.

19. $\log_{10} \sqrt[4]{48}$.

20. $\log_{10} 32$.

21. $\log_{10} 81$.

22. $\log_{10} 144$.

67. Common Logarithms. There are two systems of logarithms which are used extensively in mathematics. In one of these the base is an irrational number, designated by e, whose value is approximately 2.718. Logarithms to this base are called **natural,** or **Naperian,** logarithms. This system is convenient in many theoretical considerations and is used in calculus and higher mathematics. Logarithms to the base 10 are called the **common,** or **Briggs,** system of logarithms. Common logarithms are especially suited for making computations.

We shall be interested principally in logarithms to the base 10; and henceforth, for brevity, this base will not be indicated in logarithmic expressions. Thus we shall write $\log_{10} N$ as $\log N$.

The common logarithms of integral powers of 10 are integers. This is illustrated in the following list of powers of 10 and their logarithms.

$$\begin{aligned}
10^3 &= 1000 & \log 1000 &= 3\\
10^2 &= 100 & \log 100 &= 2\\
10^1 &= 10 & \log 10 &= 1\\
10^0 &= 1 & \log 1 &= 0\\
10^{-1} &= 0.1 & \log 0.1 &= -1\\
10^{-2} &= 0.01 & \log 0.01 &= -2\\
10^{-3} &= 0.001 & \log 0.001 &= -3
\end{aligned}$$

Noticing that $\log 1 = 0$, we may conclude that numbers greater than 1 have positive logarithms and that numbers between 0 and 1 have negative logarithms. Further, we conclude that a number between two consecutive integral powers of 10 has its logarithm between the exponents of 10 which express those powers. That is, the logarithm of a number between 10^n and 10^{n+1} is between the integers n and $n + 1$, or n plus a decimal (between 0 and 1). These facts are illustrated in the table below, where the symbol $<$ means "is less than".

$$\begin{aligned}
10^1 &< 63 < 10^2 & \log 63 &= 1 + \text{a decimal}\\
10^0 &< 8 < 10^1 & \log 8 &= 0 + \text{a decimal}\\
10^{-1} &< 0.7 < 10^0 & \log 0.7 &= -1 + \text{a decimal}\\
10^{-2} &< 0.04 < 10^{-1} & \log 0.04 &= -2 + \text{a decimal}
\end{aligned}$$

68. The Characteristic and Mantissa. When the common logarithm of a number is written as an integer plus a decimal (between 0 and 1), the integer is called the **characteristic** and the decimal part is called the **mantissa.** The logarithm of an integral power of 10 is an integer; the logarithms of other numbers have nonzero mantissas. Finding the logarithm of a number requires, of course, the determination of both the characteristic and the mantissa. We shall consider the characteristic first and discover a simple rule which is applicable to numbers in decimal notation. To state the rule, we introduce the idea of **reference position** of the decimal point.

The decimal point is in the reference position when placed immediately after the first nonzero digit of the number.

This means that numbers between 1 and 10 have the decimal point in the reference position. The decimal point in 4.832, for example, is in the reference position. The decimal point in 0.004832 is three places to the left, and in 4832 three places to the right, of the reference position. Each of these numbers, however, can be expressed as a product of 4.832 and an integral power of 10. Thus

$$0.004832 = 4.832 \times 10^{-3}$$

$$4832 = 4.832 \times 10^{3}.$$

These equations may be verified by multiplying the two factors in the right members. Multiplying a number by 10^3 merely moves the decimal point three places to the right, and multiplying by 10^{-3} moves the decimal point three places to the left.

To see what change is produced in the logarithm of a number by moving the decimal point, let us consider the numbers N, 10^nN, and $10^{-n}N$, where N is between 1 and 10 and n is a positive integer. The decimal point of N is in the reference position; the decimal point of 10^nN (when multiplied) is n places to the right of the reference position; and the decimal point of $10^{-n}N$ is n places to the left of the reference position. Since N is between 1 and 10, its logarithm is between 0 and 1. Hence, applying Laws I and III, we write

$$\begin{aligned} \log N &= 0 + \text{a decimal}, \\ \log 10^nN &= \log 10^n + \log N \\ &= n + \text{same decimal}, \\ \log 10^{-n}N &= \log 10^{-n} + \log N \\ &= -n + \text{same decimal}. \end{aligned}$$

The logarithms of these numbers have the same mantissa but different characteristics. Hence the mantissa of the logarithm of a number is independent of the location of the decimal point. On the other hand, the characteristic depends only on the location of the decimal point. The characteristics of the logarithms of N, 10^nN, and $10^{-n}N$ are, respectively, 0, n and $-n$. These results enable us to state a rule for determining characteristics.

RULE. *If the decimal point of a number is in the reference position, the characteristic of its logarithm is zero. If the decimal point is n places to the right of the reference position, the characteristic is n. If the decimal point is n places to the left of the reference position, the characteristic is $-n$.*

Example. Given $\log 5.826 = 0.7654$, write the logarithms of 582.6, 58260, and 0.05826.

Solution. The mantissas of the logarithms of these numbers are all the same as the given mantissa. Relative to the reference position, the decimal points of 582.6, 58260, and 0.05826 are, respectively, 2 places to the right, 4 places to the right, and 2 places to the left. Hence the required characteristics are 2, 4, and -2, and we have

$$\log 582.6 = 2.7654,$$

$$\log 58260 = 4.7654,$$

$$\log 0.05826 = -2 + 0.7654.$$

We have noticed that the logarithm of a number between 0 and 1 is negative. That is, the characteristic is a negative integer, and the mantissa is either zero or a positive fraction. The characteristic may be combined with the mantissa to yield a single negative quantity. For example,

$$\log 0.05826 = -2 + 0.7654 = -1.2346.$$

For most computations, however, it is preferable to express a logarithm with the fractional part positive.

The logarithm $-2 + 0.7654$ could not be written as -2.7654 since this would indicate that the fractional part is also negative. A negative characteristic is customarily expressed as the difference of two integers. Thus, since $-2 = 2 - 4$ and also equal to $8 - 10$, we have

$$\begin{aligned} \log 0.05826 &= -2 + 0.7654 \\ &= 2.7654 - 4 \\ &= 8.7654 - 10. \end{aligned}$$

A negative characteristic may obviously be written as the difference of two integers in an unlimited number of ways. The usual practice, however, is to use a positive integer minus 10 or an integral multiple of 10.

Exercise 44

Write the characteristic of the logarithm of each of the following:

1. 38.
2. 50.
3. 824.3.
4. 7935.
5. 400.1.
6. 43780.
7. 0.923.
8. 0.0372.
9. 0.00347.
10. 0.0006.
11. 0.00008.
12. 1.800.
13. 6.43×10^3.
14. 2.78×10^{-2}.
15. 7.85×10^{-5}.

Given log 86.34 = 1.9362, find the numbers which have the following logarithms:

16. 2.9362.
17. 0.9362.
18. 3.9362.
19. 8.9362 − 10.
20. 9.9362 − 10.
21. 5.9362.
22. 4.9362.
23. 18.9362 − 20.
24. 27.9362 − 30.
25. 6.9362 − 10.
26. 5.9362 − 10.
27. 16.9362 − 20.

69. Tables of Logarithms. We have shown how to determine immediately the characteristic of the logarithm of a number. Finding mantissas, however, is not simple. But methods are developed in more advanced mathematics which permit the computation of a mantissa to any desired number of decimal places. The mantissas corresponding to many numbers have been computed and arranged in tabular form. Such an arrangement is called a **table of logarithms,** or a **table of mantissas.** Since most mantissas are unending decimal fractions, their values are given only approximately. The tables, therefore, are called *three-place, four-place, five-place,* and so on, depending on the number of places used. A table may be used both for finding mantissas and for finding numbers corresponding to given mantissas.

The number corresponding to a given logarithm is called the **antilogarithm** (usually written "antilog"). That is, if $\log N = x$, then antilog $x = N$.

A four-place table of logarithms is found on pages 302 and 303. The mantissas are in the ten columns headed by the integers 0 to 9. The decimal point is omitted in each mantissa and must be supplied. We shall explain the use of the table with illustrative examples.

Example 1. Find log 0.00246.

Solution. The decimal point is three places to the left of the reference position, and this makes the characteristic −3. Although the characteristic is determined from the position of the decimal point, the mantissa is independent of the decimal point. Hence the numbers 0.00246 and 246 have the same mantissa. We now find the mantissa corresponding to 246. Look on page 302 in the first column, which has N at the top. Go down the column to 24, the first two digits in 246. Then look along the line from 24, stopping at the column which has 6 at the top, the third digit in 246.

The mantissa .3909 is thus located. Hence

$$\log 0.00246 = 7.3909 - 10.$$

Example 2. Find log 87,000.

Solution. Since the decimal point is four places to the right of the reference position, the characteristic is 4. The mantissa, found in the table on page 303, may be located by looking across from the number 87, which is in the column headed by N, to the column headed by 0. The mantissa is .9395. Hence

$$\log 87{,}000 = 4.9395.$$

Example 3. Find antilog 3.6284.

Solution. Here we have the logarithm and wish to find the corresponding number, or antilogarithm. Since the table has only the mantissas of logarithms, we disregard the given characteristic for the moment and look in the columns of mantissas for 6284. The mantissas increase to the right along each row, and from row to row. It is easy, therefore, to locate any mantissa which appears in the table. Having found the entry 6284, we look along its row to the left and find 42 in the column headed by N. The third digit is at the top of the column containing the given mantissa and, in our example, is 5. Hence the sequence of digits in 425 corresponds to the mantissa .6284. To place the decimal point, we notice that the given characteristic is 3; consequently the point belongs three places to the right of the reference position. Hence

$$\text{antilog } 3.6284 = 4250.$$

Example 4. If $\log N = 8.9741 - 10$, find N.

Solution. Again we have a logarithm and wish to find the antilogarithm. We first locate 9741 in a column of mantissas. Then along the row of this entry we find 94 in the first column of the page. The given mantissa has 2 at the top of its column. Hence N, except for the decimal point, is 942. The given characteristic is $8 - 10$, or -2, and therefore the decimal point goes two places to the left of the reference position. Accordingly, we have $N = 0.0942$.

Exercise 45

Find the logarithm of each of the following numbers:

1. 3.14.
2. 57.2.
3. 4310.
4. 57,600.
5. 9.09.
6. 0.821.
7. 0.008.
8. 0.058.
9. 0.005.
10. 760.
11. 900.
12. 0.000087.
13. 0.39.
14. 576,000.
15. 0.0802.

Find the antilogarithm of each of the following logarithms:

16. 1.1847.
17. 2.4900.
18. 3.6064.
19. 0.9708.
20. 4.7324.
21. 0.8927.
22. $9.6767 - 10$.
23. $8.6513 - 10$.
24. $7.0414 - 10$.
25. $6.8476 - 10$.
26. 0.0043.
27. $9.4843 - 10$.
28. 3.000.
29. 1.9996.
30. $7.8692 - 10$.

70. Interpolation. In finding logarithms and antilogarithms in the preceding section only direct readings from the table were necessary. This was because logarithms of numbers of three or fewer digits (initial and final zeros excluded) were required; and, in the case of a given logarithm, the mantissa appeared in the table. The determination of the logarithm of a four-digit number is not so readily obtained. Also, a given mantissa which is not in the table presents a slight difficulty. These cases can be handled, however, by the use of a process called **interpolation.** According to the principle of interpolation, $\log N$ and N are directly proportional for small changes in N. While this assumption is not strictly true, its use yields satisfactory approximations.

In the interpolation process we shall have occasion to **round off** numbers, that is, to discard extra digits. In each case the new number will be an approximation to the given number.

Illustrations:

We round off 2.718 to three digits and have 2.72. We add 1 to the third digit of the given number in order to obtain the best three-figure approximation. For two-figure and one-figure approximations we round off respectively to 2.7 and 3.

In rounding off 2.715 to three digits, we observe that 2.71 and 2.72 are equally valid. In all such cases we shall arbitrarily **round off so that the last digit is an even number.** Accordingly we choose 2.72.

Example 1. Find log 2047.

Solution. The number 2047 is 0.7 of the way from 2040 to 2050. The mantissas corresponding to 2040 and 2050 are the same as those for 204 and 205 and can be read directly from the table. These mantissas are 3096 and 3118, except for the decimal points. To obtain a number 0.7 of the way from the smaller to the larger of these numbers, we compute $0.7(3118 - 3096)$. Our answer is 15, to the nearest integer, and hence $3096 + 15 = 3111$ is the required mantissa when the decimal point is supplied. The quantities involved in this interpolation process are tabulated as a further aid in understanding the steps.

$$\begin{array}{lcl}
\begin{array}{cc} & \text{Number} \\ 10 & \left[\begin{array}{l} 7\left[\begin{array}{l}2040\\2047\end{array}\right. \\ \,2050 \end{array}\right. \end{array}
&
\begin{array}{c} \text{Mantissa} \\ \left.\begin{array}{r}3096\\?\\3118\end{array}\right]22 \end{array}
&
\begin{array}{r} 0.7 \times 22 = 15.4 \\ = 15\text{, rounded off} \\ 3096 + 15 = 3111 \end{array}
\end{array}$$

Since the required characteristic is 3, we have

$$\log 2047 = 3.3111.$$

Example 2. Find log 0.07542.

Solution. The characteristic is -2, or $8 - 10$. To find the mantissa, we recall that this part of the logarithm of a number is independent of the location of the decimal point. Hence we find the mantissa corresponding to the digits in 7542. The quantities involved are placed here in tabular form.

$$\begin{array}{lcl}
\begin{array}{cc} & \text{Number} \\ 10 & \left[\begin{array}{l} 2\left[\begin{array}{l}7540\\7542\end{array}\right. \\ \,7550 \end{array}\right. \end{array}
&
\begin{array}{c} \text{Mantissa} \\ \left.\begin{array}{r}8774\\?\\8779\end{array}\right]5 \end{array}
&
\begin{array}{r} 0.2 \times 5 = 1.0 \\ \\ 8774 + 1 = 8775 \end{array}
\end{array}$$

We see that 7542 is 0.2 of the way across from 7540 to 7550; and we want, therefore, to find the number which is 0.2 of the way from 8774 to 8779. From 8774 to 8779 is 5 units, and 0.2 of 5 is 1. Hence $8774 + 1 = 8775$, and

$$\log 0.07542 = 8.8775 - 10.$$

The student who is just beginning to learn the method of interpolation will probably need to tabulate the quantities in a way similar to that used in the preceding examples. Having done a few problems, however, he should notice that most, if not all, of the operations can be done mentally. As shown in the examples, three operations are involved in finding a mantissa.

1. A mantissa is to be subtracted from the one immediately following in the table.
2. This difference is to be multiplied by an integral multiple of tenths (0.1 or more up to 0.9).
3. The product, rounded off, is to be added to the smaller mantissa.

In most cases the mental performance of these operations can be made easily and quickly.

Let us next consider the problem of finding the number corresponding to a given logarithm whose mantissa does not appear in the table. This is the inverse of the problems just done, but the principle of interpolation is the same.

Example 3. Find the antilog 7.9343 − 10.

Solution. The mantissa 9343 is between the consecutive mantissas 9340 and 9345, and is $\frac{3}{5}$ of the way across. The numbers 8590 and 8600 correspond to the consecutive mantissas. Hence we find the number $\frac{3}{5}$ of the way from 8590 to 8600. This gives 8596 which, except for the decimal point, is the required antilogarithm. Placing the decimal point in accordance with the given characteristic 7 − 10, or −3, we obtain

$$\text{antilog } 7.9343 - 10 = 0.008596.$$

Again, the student should observe that, in most cases, the computations required in finding antilogarithms can be done mentally.

Exercise 46

Find the logarithms of the following numbers:

1. 68.25.
2. 7.831.
3. 585.3.
4. 0.9872.
5. 0.09864.
6. 0.004446.
7. 64,050.
8. 99.09.
9. 4.827.
10. 0.001118.
11. 0.01001.
12. 0.00007447.
13. 90490.
14. 10.51.
15. 234.8.
16. 0.8105.
17. 0.0003077.
18. 0.4805.
19. 1741.
20. 7621.
21. 7897.
22. 98,890.
23. 745,500.
24. 0.2604.

Find the antilogarithms of the following logarithms:

25. 2.7743.
26. 0.8104.
27. 1.8063.
28. 3.0242.
29. 4.9985.
30. 0.6090.
31. 7.6621 − 10.
32. 8.4787 − 10.
33. 9.4885 − 10.
34. 9.1336 − 10.
35. 8.1731 − 10.
36. 6.3011 − 10.
37. 0.2330.
38. 1.2479.
39. 2.3182.
40. 3.3200.
41. 4.3346.
42. 5.2901.
43. 9.7375 − 10.
44. 8.6920 − 10.
45. 7.9511 − 10.
46. 8.0100 − 10.
47. 7.0350 − 10.
48. 9.6626 − 10.

71. Logarithms Used in Computations. We are now ready to compute products, quotients, powers, and roots, or any combination of these, by means of logarithms. The three laws of logarithms (Section 66) serve for this purpose. We shall illustrate their use in some examples.

Example 1. Compute N by means of logarithms, given

$$N = \frac{(32.41)(81.93)}{(1.854)(0.7949)}.$$

Solution. Using Law II, we write

$$\log N = \log (32.41)(81.93) - \log (1.854)(0.7949).$$

And by Law I,

$$\log N = \log 32.41 + \log 81.93 - (\log 1.854 + \log 0.7949).$$

To help in avoiding errors and also to speed the work, the student should first make an outline form in which the characteristics of the given numbers are written and spaces are left for the quantities to be found. We suggest a form for this problem:

$$\begin{array}{rl@{\qquad\qquad}rl}
\multicolumn{2}{c}{\textit{Numerator}} & \multicolumn{2}{c}{\textit{Denominator}} \\
\log 32.41 = & 1. & \log 1.854 = & 0. \\
\log 81.93 = & 1. & \log 0.7949 = & 9. \qquad\qquad - 10 \\
\cline{2-2}\cline{4-4}
\log \text{num} = & & \log \text{denom} = & \\
\log \text{denom} = & & & \\
\cline{2-2}
\log N = & & &
\end{array}$$

Having made the form, the tables should then be used and the spaces filled in. We show the completed form:

$$\begin{array}{rl@{\qquad\qquad}rl}
\log 32.41 = & 1.5106 & \log 1.854 = & 0.2681 \\
\log 81.93 = & 1.9135 & \log 0.7949 = & 9.9003 - 10 \\
\cline{2-2}\cline{4-4}
\log \text{num} = & 3.4241 & \log \text{denom} = & 10.1684 - 10 \\
\log \text{denom} = & 0.1684 & = & 0.1684 \\
\cline{2-2}
\log N = & 3.2557 & & \\
N = & 1802. & &
\end{array}$$

Example 2. Evaluate $\sqrt[3]{321.4}/\sqrt{208.7}$.
Solution. Letting N stand for the given fraction, we have

$$N = \frac{\sqrt[3]{321.4}}{\sqrt{208.7}} = \frac{(321.4)^{1/3}}{(208.7)^{1/2}},$$

and
$$\log N = \tfrac{1}{3}\log 321.4 - \tfrac{1}{2}\log 208.7.$$

$$\log 321.4 = 2.5071 \qquad \log 208.7 = 2.3195$$

$$\tfrac{1}{3}\log 321.4 = 0.8357 \qquad \tfrac{1}{2}\log 208.7 = 1.1598$$

The logarithm of the denominator is greater than the logarithm of the numerator. We wish to obtain the difference, or log N, in the form of a negative integer plus a positive fraction (mantissa). The table may then be used to find N. To obtain log N in this usable form, we first express the characteristic of the logarithm of the numerator as 10 − 10. Thus we write

$$\begin{aligned} \tfrac{1}{3}\log 321.4 &= 10.8357 - 10 \\ \tfrac{1}{2}\log 208.7 &= 1.1598 \\ \hline \log N &= 9.6759 - 10 \\ N &= 0.4741. \end{aligned}$$

Example 3. Compute $\sqrt[3]{0.8316}$.

Solution. We first write $N = \sqrt[3]{0.8316} = (0.8316)^{1/3}$. Then

$$\log N = \tfrac{1}{3}\log 0.8316$$

$$\log 0.8316 = 9.9199 - 10.$$

Dividing 9.9199 − 10 by 3, we get the result, 3.3066 − 3.3333, in an undesirable form. A usable form is obtained at once if the characteristic 9 − 10 is replaced by the equal quantity 29 − 30. Thus we get

$$\begin{aligned} \log 0.8316 &= 29.9199 - 30 \\ \tfrac{1}{3}\log 0.8316 &= 9.9733 - 10 \\ N &= 0.9404. \end{aligned}$$

Exercise 47

Compute by means of logarithms:

1. (3.82)(61.2).
2. (5.13)(0.837).
3. (0.00217)(622).
4. (47.6)(0.0348).
5. (0.631)(7.11).
6. (0.0081)(723).
7. (0.9093)(0.3909).
8. (87.17)(1.093).
9. (771.3)(31.25).
10. 44.8 ÷ 81.3.
11. 224 ÷ 68.7.
12. 9.07 ÷ 75.6.
13. 8034 ÷ 8307.
14. 81.81 ÷ 1.194.
15. 3333 ÷ 566.5.
16. 0.7714 ÷ 7.704.
17. 55.66 ÷ 1.194.
18. 45.67 ÷ 1.083.
19. (5.01)(3.77)(9.12).
20. (10.7)(8.16)(70.7).
21. (4.771)(8.224)(3.016).
22. (90.92)(0.141)(6.683).
23. (7.103)(23.09)(1.783).
24. (12.36)(357.9)(0.167).

25. $\dfrac{(314)(671)}{(544)(837)}$.

26. $\dfrac{(404)(505)}{(707)(114)}$.

27. $\dfrac{(7531)(3456)}{(1813)(1339)}$.

28. $\dfrac{(0.01843)(8.711)}{(0.4622)(91.27)}$.

29. $\dfrac{(1.743)(43.12)}{(347.1)(5248)}$.

30. $\dfrac{(9.714)(7.941)}{(3743)(3.842)}$.

31. $(2.718)^2$.

32. $(3.318)^3$.

33. $(1.147)^4$.

34. $(0.8703)^3$.

35. $(0.9718)^2$.

36. $(0.07184)^4$.

37. $\sqrt{6843}$.

38. $\sqrt{518.3}$.

39. $\sqrt{0.6118}$.

40. $\sqrt[3]{75.66}$.

41. $\sqrt[3]{8.082}$.

42. $\sqrt[3]{66.78}$.

43. $\sqrt[4]{0.04623}$.

44. $\sqrt[4]{17.81}$.

45. $\sqrt[5]{64.87}$.

46. $(21.08)^{2/3}$.

47. $(0.1776)^{3/4}$.

48. $(1.414)^{3/5}$.

49. $(11.08)^{-4/3}$.

50. $(1881)^{-3/5}$.

51. $(1004)^{-5/2}$.

52. $(0.3284)^{1/5}$.

53. $(0.2875)^{1/6}$.

54. $(7531)^{1/7}$.

55. $\sqrt[3]{8143}/\sqrt{5625}$.

56. $\sqrt{56.21}/\sqrt[3]{65.12}$.

57. $\sqrt{1357}/(5.634)^3$.

58. $25.73\sqrt{5.183}/\sqrt[3]{893.2}$.

59. $\sqrt[3]{800.4}/1439\sqrt{1.934}$.

60. $\sqrt[3]{\dfrac{62.13}{1.783}}$.

61. $\sqrt{\dfrac{(57.81)(6.183)}{19.78}}$.

62. $\sqrt[3]{\dfrac{(4881)(2321)}{87.91}}$.

63. $\sqrt[4]{\dfrac{(19.03)(30.91)}{(1.456)(2.531)}}$.

72. Exponential Equations.

An equation in which the unknown appears in an exponent is called an **exponential equation.** Some quite simple equations of this kind may be solved by inspection. Certain others are readily solved by the use of logarithms. The following examples illustrate methods of solving exponential equations.

Example 1. Solve $5^x = 625^2$.

Solution. First express 625 as 5^4, then $5^x = (5^4)^2 = 5^8$. With the bases equal, the exponents must be the same; hence, $x = 8$.

Example 2. Solve $2^x = 29$.

Solution. Since equal quantities have equal logarithms, we may write

$$\log 2^x = \log 29.$$

Whence,

$$x \log 2 = \log 29,$$

and

$$x = \frac{\log 29}{\log 2} = \frac{1.4624}{0.3010} = 4.858.$$

The value of x is obtained by dividing log 29 by log 2. This is the quotient of two logarithms; it should not be confused with the logarithm of a quotient.

Example 3. Solve the equation $5(3)^{2x+1} = 7^{1-x}$.

Solution. Taking the logarithm of each member gives

$$\log 5(3)^{2x+1} = \log 7^{1-x},$$

and

$$\log 5 + (2x + 1) \log 3 = (1 - x) \log 7.$$

This is a linear equation in x and may be solved in the usual way. Thus

$$(2 \log 3 + \log 7)x = \log 7 - \log 5 - \log 3$$

and

$$x = \frac{\log 7 - \log 5 - \log 3}{2 \log 3 + \log 7} = \frac{-0.3310}{1.7993} = -0.1840.$$

Exercise 48

Solve the following exponential equations without the use of tables.

1. $5^x = 125.$
2. $3^{x-1} = 9.$
3. $e^{2x} = e^{10}.$
4. $4^{x+2} = 16^x.$
5. $10^{3x-3} = 1000.$
6. $25^x = 5.$
7. $4^{2x+3} = 4.$
8. $6^x = \frac{1}{36}.$
9. $3^{2x} = 3^0.$

Solve the following exponential equations:

10. $4^x = 7.$
11. $9^x = 15.$
12. $(1.27)^x = 2.73.$
13. $9^x = 7.$
14. $6^{2x-3} = 4.$
15. $(8.13)^x = 68.$
16. $4^x = 5^x.$
17. $11^{2x-1} = 7^x.$
18. $20^x = 4^x.$
19. $100^x = 3^x.$
20. $(0.3)^x = 0.5.$
21. $(0.182)^x = 0.93.$
22. $(0.718)^x = 1.07.$
23. $3^x = 0.13.$
24. $17^x = 0.19.$
25. $6 \cdot 10^x = 11^x.$
26. $9^x = 4 \cdot 19^{x+1}.$
27. $5 \cdot 7^x = 6 \cdot 8^x.$

73. Logarithms of a Number to Different Bases. It is sometimes desirable to express the logarithm of a number to one base in terms of its logarithm to another base. In particular, a change from a common logarithm to a natural logarithm, or vice versa, is advantageous in certain calculus problems. We shall derive the relation between the logarithms of a number to any two bases.

Let a and b stand for two positive numbers different from 1, and let

$$\log_a N = x \quad \text{or} \quad N = a^x.$$

Taking the logarithm to the base b of both members of the exponential equation, we obtain

$$\log_b N = x \log_b a$$

and

$$x = \frac{\log_b N}{\log_b a}.$$

Since $x = \log_a N$, we obtain

$$\log_a N = \frac{\log_b N}{\log_b a}.$$

This formula can be used to change from one base to another. By substituting the natural base e (2.718 approximately) for a and 10 for b, we have

$$\log_e N = \frac{\log_{10} N}{\log_{10} e}.$$

The value of $\log_{10} e$ is 0.4343 to four decimal places. Making this substitution, we get the relations

$$\mathbf{\log_{10} N = 0.4343 \log_e N}$$

and

$$\mathbf{\log_e N = 2.3026 \log_{10} N.}$$

Illustrations:

$$\log_e 46 = 2.3026 \log_{10} 46 = (2.3026)(1.6628) = 3.829.$$

$$\log_4 15 = \frac{\log_{10} 15}{\log_{10} 4} = \frac{1.1761}{0.6021} = 1.953.$$

Exercise 49

Find the following logarithms:

1. $\log_e 100$.
2. $\log_e 158$.
3. $\log_e 0.478$.
4. $\log_2 54$.
5. $\log_3 175$.
6. $\log_4 204$.
7. $\log_2 0.416$.
8. $\log_5 0.067$.
9. $\log_7 62.4$.
10. $\log_{11} 150$.
11. $\log_e 0.041$.
12. $\log_{12} 247$.
13. $\log_8 9.85$.
14. $\log_9 0.009$.
15. $\log_e \sqrt{2}$.

Chapter Ten

QUADRATIC EQUATIONS

74. Quadratic Equations in One Unknown. An equation which can be expressed in the form

$$ax^2 + bx + c = 0,$$

where a, b and c are constants and $a \neq 0$, is called a **quadratic equation** or an **equation of the second degree** in x. If a were zero, the x^2 term would be lacking and the equation would not be of the second degree. The equation, as here written, is said to be in **standard form** because all the nonzero terms are in the left member and appear in the order of descending powers of the unknown.

If the first power of the unknown is missing in a quadratic equation (that is, when $b = 0$), the equation is called an **incomplete,** or a **pure, quadratic equation.** If the first degree term is present, the equation is called a **complete quadratic equation.**

Illustrations:

$7x^2 - 4 = 0$ is a pure quadratic equation.

$6x^2 - 5x + 4 = 0$ is a complete quadratic equation.

An important operation in connection with a quadratic equation is that of finding the solution, or roots, of the equation. We shall consider this problem in detail. We note in advance, however, that the roots of a pure quadratic equation can be readily and simply obtained.

Example 1. Solve the equation $4x^2 - 9 = 0$.

Solution. We solve for x^2 and then take square roots. Thus

$$4x^2 = 9,$$

$$x^2 = \tfrac{9}{4},$$

$$x = \pm\sqrt{\tfrac{9}{4}} = \pm\tfrac{3}{2}.$$

The double sign means that the roots are the two numbers $+\frac{3}{2}$ and $-\frac{3}{2}$. Each of these values satisfies the given equation.

Example 2. Solve the equation $9x^2 + 4 = 0$.

Solution.

$$9x^2 = -4,$$

$$x^2 = -\tfrac{4}{9},$$

$$x = \pm\sqrt{-\frac{4}{9}} = \pm i\sqrt{\frac{4}{9}} = \pm\frac{2i}{3}.$$

75. Solution by Factoring. If the left member of a quadratic equation in standard form can be separated into two linear factors, the roots can be had at once from the factors. The key principle involved in this method is the fact that the product of two factors is zero if either of the factors is zero. Thus, $A \cdot B = 0$ if $A = 0$ or if $B = 0$. Accordingly, if the left member of the equation is factored, a value of the unknown which makes either factor equal to zero makes the product of the factors zero and therefore is a root. Hence the two linear equations obtained by equating each factor to zero yield the roots.

Example 1. Solve the equation $6x^2 + 5x = 4$.

Solution. The equation as given is not in standard form. Hence we first transpose 4 to the left side. This gives the equivalent equation

$$6x^2 + 5x - 4 = 0.$$

Factored form:

$$(2x - 1)(3x + 4) = 0.$$

Set each factor equal to zero:

$$2x - 1 = 0, \qquad 3x + 4 = 0,$$

$$x = \tfrac{1}{2}. \qquad x = -\tfrac{4}{3}.$$

Check:

$$6(\tfrac{1}{2})^2 + 5(\tfrac{1}{2}) = 4, \qquad 6(-\tfrac{4}{3})^2 + 5(-\tfrac{4}{3}) = 4,$$

$$\tfrac{6}{4} + \tfrac{5}{2} = 4, \qquad \tfrac{32}{3} - \tfrac{20}{3} = 4,$$

$$\frac{6 + 10}{4} = 4, \qquad \frac{32 - 20}{3} = 4,$$

$$4 = 4. \qquad 4 = 4.$$

Example 2. Solve the equation $-3x^2 - 5x + 12 = 0$.

Solution. For convenience in factoring we write the equation with the coefficient of the squared term positive. Thus we obtain

$$3x^2 + 5x - 12 = 0,$$

$$(x + 3)(3x - 4) = 0.$$

The linear factors yield the roots -3 and $\frac{4}{3}$, which could be checked in the original equation.

Exercise 50

Solve the following pure quadratic equations:

1. $x^2 - 9 = 0$.
2. $4x^2 - 81 = 0$.
3. $4x^2 = 100$.
4. $6x^2 - 24 = 0$.
5. $7x^2 - 63 = 0$.
6. $16x^2 = 81$.
7. $x^2 - 3 = 0$.
8. $x^2 - a^2 = 0$.
9. $2x^2 = 3$.
10. $a^2x^2 + b^2 = 0$.
11. $x^2 + 9 = 0$.
12. $x^2 + 16 = 0$.
13. $4x^2 + 49 = 0$.
14. $x^2 + 7 = 0$.
15. $9x^2 + 2 = 0$.
16. $4x^2 + 1 = 0$.
17. $2x^2 + 3 = 0$.
18. $5x^2 + 4 = 0$.

Solve each of the following equations for x by the factoring method. Check your solutions.

19. $x^2 - x - 2 = 0$.
20. $x^2 + x - 6 = 0$.
21. $x^2 - 2x = 0$.
22. $x^2 = 7x - 12$.
23. $x^2 + 9x + 8 = 0$.
24. $3x^2 + 5x = 0$.
25. $2x^2 - x = 10$.
26. $6x^2 + 11x = -3$.
27. $7x^2 - 11x = 0$.
28. $8x^2 + 6x + 1 = 0$.
29. $5x^2 = 13x + 6$.
30. $4x^2 - 49 = 0$.
31. $10x^2 = 7x + 12$.
32. $7x^2 - 9x + 2 = 0$.
33. $36x^2 - 1 = 0$.
34. $6x^2 - 5x - 6 = 0$.
35. $4x^2 + 9 = 15x$.
36. $a^2x^2 - 9 = 0$.
37. $x^2 + 3a^2 = 4ax$.
38. $a^2x^2 + ax = 2$.
39. $9x^2 + 4x = 0$.
40. $x^2 + 4x + 4 = 0$.
41. $x^2 - 6x + 9 = 0$.
42. $b^2x^2 - a^2 = 0$.

43. $x^2 + (b + 1)x + b = 0$.
44. $x^2 - ax - bx + ab = 0$.
45. $a^2x^2 - abx - 2b^2 = 0$.
46. $ax^2 - ax - bx + b = 0$.
47. $abx^2 + a^2x - b^2x - ab = 0$.
48. $3ax^2 + abx - 3bx - b^2 = 0$.

76. Solution by Completing the Square. The solution of a quadratic equation is best obtained by factoring if the factoring can be done readily. In other cases a general method, called **completing the square,** may be used. An essential step in this method is the addition of a term to a binomial of the form $x^2 + bx$ to make a trinomial which is a perfect square. The term to be added is the square of one-half the coefficient of x. We have then

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2.$$

Example 1. Solve the equation $3x^2 + 5x - 2 = 0$.

Solution. We first transpose -2 and have

$$3x^2 + 5x = 2.$$

Divide by 3:

$$x^2 + \tfrac{5}{3}x = \tfrac{2}{3}.$$

One-half the coefficient of x is $\frac{5}{6}$. Hence we add $(\frac{5}{6})^2 = \frac{25}{36}$ to both sides. This gives

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{25}{36} + \frac{2}{3} = \frac{25 + 24}{36},$$

or

$$(x + \tfrac{5}{6})^2 = \tfrac{49}{36}.$$

Take square roots:

$$x + \tfrac{5}{6} = \pm\tfrac{7}{6},$$

$$x = -\frac{5}{6} \pm \frac{7}{6} = \frac{-5 \pm 7}{6},$$

$$x = \tfrac{1}{3} \text{ and } -2.$$

The roots $\frac{1}{3}$ and -2 can be checked in the given equation.

Example 2. Solve $2x^2 + 3x + 4 = 0$.

Solution. Transpose the constant term:

$$2x^2 + 3x = -4.$$

Divide by 2:

$$x^2 + \tfrac{3}{2}x = -2.$$

Add $(\frac{3}{4})^2$:

$$x^2 + \frac{3}{2}x + \frac{9}{16} = \frac{9}{16} - 2 = \frac{9 - 32}{16} = -\frac{23}{16},$$

or

$$\left(x + \frac{3}{4}\right)^2 = \frac{-23}{16}.$$

Take square roots:

$$x + \frac{3}{4} = \frac{\pm\sqrt{-23}}{4},$$

$$x = -\frac{3}{4} \pm \frac{\sqrt{-23}}{4} = -\frac{3}{4} \pm \frac{\sqrt{23}}{4}i.$$

In each of the preceding examples we used the double sign to indicate both square roots of the right side. It may be observed that the double sign on the left side would yield no additional values for x.

The process of solving a quadratic equation by completing the square, where x stands for the unknown, comprises the following steps:

1. Place the x^2 and x terms on the left side of the equation and the constant term or terms on the right side.

2. Divide by the coefficient of x^2.

3. Square one-half the coefficient of x and add to both sides. This makes the left side a perfect square.

4. Take the square root of both sides, using the double sign on the right side. Solve the two resulting linear equations for x.

Exercise 51

Solve for x by completing the square, and check:

1. $x^2 + 2x - 3 = 0.$
2. $x^2 - 5x + 6 = 0.$
3. $x^2 + 4x + 3 = 0.$
4. $x^2 + x - 20 = 0.$
5. $x^2 + 7x + 12 = 0.$
6. $x^2 + x - 30 = 0.$
7. $2x^2 - x - 3 = 0.$
8. $2x^2 = 5x - 2.$
9. $6x^2 + 1 = 5x.$
10. $8x^2 - 6x + 1 = 0.$
11. $6x^2 = 5x + 6.$
12. $10x^2 = 6 - 9x.$
13. $x^2 - 2x - 1 = 0.$
14. $x^2 + 4x + 1 = 0.$
15. $x^2 - 8x = 1.$
16. $x^2 - 6x = 3.$
17. $2x^2 + 4x + 1 = 0.$
18. $4x^2 - 11 = 4x.$
19. $9x^2 + 11 = 24x.$
20. $2x^2 + 21 = 14x.$
21. $3x^2 = 4x + 2.$
22. $25x^2 = 10x + 17.$
23. $16x^2 + 1 = 24x.$
24. $x^2 - 2x + 2 = 0.$
25. $x^2 + 6x + 13 = 0.$
26. $x^2 - 8x + 25 = 0.$
27. $x^2 - 2x + 3 = 0.$
28. $x^2 - 3x + 3 = 0.$
29. $9x^2 + 10 = 12x.$
30. $2x^2 = 10x - 9.$
31. $2x^2 = 3ax + 2a^2.$
32. $a^2x^2 = ax + 2.$
33. $a^2x^2 = abx + 2b^2.$
34. $x^2 - 2ax + a^2 - b^2 = 0.$
35. $x^2 - ax + ab - b^2 = 0.$
36. $x^2 + (a + b)x + ab = 0.$
37. $x^2 + 2ax - 3bx - 6ab = 0.$

77. Solution by Formula. The equation $ax^2 + bx + c = 0$ is called the **general quadratic equation.** The substitution of definite numerical values for the literal numbers a, b, and c leads to a particular quadratic equation, provided, of course, the value zero is not used for a. For example, the equation $3x^2 - 5x + 7 = 0$ has $a = 3$, $b = -5$, and $c = 7$. If we solve the general quadratic equation, we obtain the roots in terms of the literal coefficients. The roots thus obtained may be used as a formula to solve any quadratic equation. We shall derive the formula by the completing of the square method. Each step should be closely followed. We start with the equation

$$ax^2 + bx + c = 0.$$

Transpose c:

$$ax^2 + bx = -c.$$

Divide by a:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add $(b/2a)^2$:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a},$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Take square roots:

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

Hence

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

This is the **quadratic formula.** It is of such great importance that both the formula and the equation from which it came should be clearly fixed in mind.

Example 1. Solve the equation $2x^2 + 7x - 4 = 0$.

Solution. Comparing this equation with the general quadratic equation, we see that $a = 2$, $b = 7$, and $c = -4$. The solution therefore may be obtained by substituting these values for a, b, and c in the quadratic formula. Thus we get

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-4)}}{2(2)} = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$= \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4} = \frac{1}{2} \text{ and } -4.$$

The roots are $\frac{1}{2}$ and -4. These values could be checked in the given equation.

Example 2. Solve the equation $3x^2 - 4x + 5 = 0$.

Solution. Here $a = 3$, $b = -4$, and $c = 5$. Hence, substituting in the formula, we find

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(5)}}{2(3)} = \frac{4 \pm \sqrt{16 - 60}}{6},$$

$$x = \frac{4 \pm \sqrt{-44}}{6} = \frac{4 \pm 2\sqrt{-11}}{6} = \frac{2 \pm i\sqrt{11}}{3}.$$

Check:

$$3\left(\frac{2+i\sqrt{11}}{3}\right)^2 - 4\left(\frac{2+i\sqrt{11}}{3}\right) + 5 = 0,$$

$$3\left(\frac{4+4i\sqrt{11}+11i^2}{9}\right) - \frac{8+4i\sqrt{11}}{3} + 5 = 0,$$

$$\frac{4+4i\sqrt{11}-11-8-4i\sqrt{11}+15}{3} = 0,$$

$$0 = 0.$$

The other root $\dfrac{2-i\sqrt{11}}{3}$ may be checked similarly, or the checking may be done together by carrying along the double signs.

Exercise 52

Solve for x by the quadratic formula, and check.

1. $x^2 + x - 6 = 0.$
2. $x^2 - x - 12 = 0.$
3. $x^2 - 6x + 8 = 0.$
4. $x^2 + 9x + 20 = 0.$
5. $x^2 - 10x + 25 = 0.$
6. $x^2 + 8x + 16 = 0.$
7. $2x^2 - 3x - 2 = 0.$
8. $3x^2 + x - 2 = 0.$
9. $4x^2 - 12x + 9 = 0.$
10. $3x^2 + 2x - 5 = 0.$
11. $20x^2 + 9x + 1 = 0.$
12. $42x^2 - x - 1 = 0.$
13. $x^2 + x - 1 = 0.$
14. $x^2 - x - 4 = 0.$
15. $x^2 - x - 3 = 0.$
16. $2x^2 + 5x + 1 = 0.$
17. $3x^2 + 10x + 4 = 0.$
18. $7x^2 + x - 5 = 0.$
19. $4x^2 - 9x + 3 = 0.$
20. $3x^2 + 11x + 7 = 0.$
21. $x^2 + x + 1 = 0.$
22. $x^2 - x + 1 = 0.$
23. $x^2 - 2x + 2 = 0.$
24. $x^2 - 6x + 10 = 0.$
25. $2x^2 - 2x + 5 = 0.$
26. $3x^2 + 5x + 1 = 0.$
27. $3x^2 - 19x + 9 = 0.$
28. $6x^2 - x + 4 = 0.$
29. $x^2 - ax - 6a^2 = 0.$
30. $6x^2 + 7ax + 2a^2 = 0.$

31. $x^2 + (b - a)x - ab = 0.$
32. $mx^2 + (1 + m)x + 1 = 0.$
33. $mx^2 + (m^2 + 1)x + m = 0.$
34. $abx^2 + (2b - 3a)x - 6 = 0.$
35. $(a + b)^2x^2 - (a + b)x - 2 = 0.$
36. $x^2 + 5x - 4m^2 + 6m + 4 = 0.$
37. $x^2 + 5ax + 6a^2 + ab - b^2 = 0.$
38. $x^2 + (m - n)x - 2(m - n)^2 = 0.$

Use the quadratic formula to solve for x in terms of y and for y in terms of x.

39. $x^2 - 2xy + y^2 + x - y = 0.$
40. $x^2 - xy - 2y^2 - 2x + 4y = 0.$
41. $2x^2 - 5xy + 3y^2 - 2x + 3y = 0.$
42. $3x^2 + 8xy - 3y^2 + 2x + 6y = 0.$
43. $3x^2 + 8xy - 3y^2 + 2x + 6y = 0.$
44. $x^2 + 2xy + y^2 + x + y = 2.$
45. $x^2 - xy - 12y^2 + 5x + 6 = 6y.$
46. $x^2 - 4xy + 4y^2 + 2x + 1 = 5y.$

78. Equations in Quadratic Form. An equation is often met which is not quadratic but can be reduced to a quadratic equation by the substitution of a new unknown. Such an equation is said to be in **quadratic form.**

For example, the fourth degree equation

$$y^4 - 3y^2 - 4 = 0,$$

when written as

$$(y^2)^2 - 3y^2 - 4 = 0,$$

is seen to be a quadratic equation in y^2. The substitution $x = y^2$ yields the quadratic equation

$$x^2 - 3x - 4 = 0.$$

The following equations are further illustrations of equations in quadratic form:

$$x^{-3} + 4x^{-3/2} + 5 = 0 \text{ is quadratic in } x^{-3/2},$$

$$(x^2 + x)^2 + 2(x^2 + x) - 1 = 0 \text{ is quadratic in } x^2 + x.$$

These equations would become quadratic equations by the substitution of a single letter for $x^{-3/2}$ in the first and for $x^2 + x$ in the second.

Some equations in quadratic form can be solved quite readily. The following examples illustrate useful methods.

Example 1. Solve the equation $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0$.

Solution. The substitution $y = x^2 + x$ reduces the equation to

$$y^2 - 2x - 8 = 0.$$

Whence

$$(y + 2)(y - 4) = 0,$$

and

$$y = -2 \text{ and } 4.$$

Since $y = x^2 + 3x$, we have the equations

$$x^2 + 3x = -2 \quad \text{and} \quad x^2 + 3x = 4.$$

We next solve these equations.

$$x^2 + 3x + 2 = 0, \qquad x^2 + 3x - 4 = 0,$$

$$(x + 2)(x + 1) = 0, \qquad (x + 4)(x - 1) = 0,$$

$$x = -2 \text{ and } -1. \qquad x = -4 \text{ and } 1.$$

The given equation has the four roots -4, -2, -1, and 1.

Example 2. Solve the equation $x + 2x^{1/2} - 8 = 0$.

Solution. This equation would be reduced to a quadratic equation by substituting a single letter for $x^{1/2}$. However, it may be readily solved by factoring. Thus, employing fractional exponents, we have

$$(x^{1/2} - 2)(x^{1/2} + 4) = 0.$$

Equating the first factor to zero, we obtain

$$x^{1/2} - 2 = 0, \quad \text{or } x^{1/2} = 2, \quad \text{and } x = 4.$$

Since $x^{1/2}$ cannot be negative, it is impossible to make the second factor $x^{1/2} + 4$ equal to zero. Hence the only solution is $x = 4$.

Example 3. Solve the equation $\dfrac{x^2 - 4}{x} - \dfrac{4x}{x^2 - 4} + 3 = 0.$

Solution. This equation, though not in quadratic form, may be reduced to a quadratic equation. Substituting $y = \dfrac{x^2 - 4}{x}$ gives

$$y - \frac{4}{y} + 3 = 0,$$

and, multiplying by y, we obtain

$$y^2 + 3y - 4 = 0.$$

The roots of this equation are $y = -4$ and 1. Hence we form the equations

$$\frac{x^2 - 4}{x} = -4 \quad \text{and} \quad \frac{x^2 - 4}{x} = 1.$$

These equations yield:

$$x^2 + 4x - 4 = 0, \qquad x^2 - x - 4 = 0,$$

$$x = \frac{-4 \pm \sqrt{16 + 16}}{2} \qquad x = \frac{1 \pm \sqrt{1 + 16}}{2}$$

$$= -2 \pm 2\sqrt{2}. \qquad = \frac{1 \pm \sqrt{17}}{2}.$$

Exercise 53

Solve the following equations:

1. $x^4 - 10x^2 + 9 = 0.$ **2.** $x^4 - x^2 - 2 = 0.$ **3.** $4x^4 - 5x^2 + 1 = 0.$
4. $2x^4 + 7x^2 - 4 = 0.$ **5.** $x^4 + 7x^2 + 12 = 0.$ **6.** $x^4 + 4x^2 + 3 = 0.$

7. $x^{-2} + x^{-1} - 12 = 0.$ **8.** $2x^{-2} + x^{-1} - 1 = 0.$
9. $3x^{-2} + 4x^{-1} - 4 = 0.$ **10.** $4x^{-2} + x^{-1} - 5 = 0.$
11. $x^{-4} - 17x^{-2} + 16 = 0.$ **12.** $x^{-4} - 8x^{-2} - 9 = 0.$
13. $x^{2/3} + 7x^{1/3} + 12 = 0.$ **14.** $x^{4/3} - 5x^{2/3} + 4 = 0.$
15. $x - 4x^{1/2} + 3 = 0.$ **16.** $x - 3x^{1/2} - 10 = 0.$
17. $6x^{1/2} - 11x^{1/4} - 2 = 0.$ **18.** $x - x^{1/2} - 30 = 0.$
19. $x^{1/2} - 3x^{-1/2} + 2 = 0.$ **20.** $6x^{1/2} + x^{-1/2} - 5 = 0.$

21. $x^2 - 3x - 3\sqrt{x^2 - 3x} + 2 = 0.$

22. $x^2 + 2x - \sqrt{x^2 + 2x} - 2 = 0.$

23. $x^2 - 9x^{-2} - 8 = 0.$

24. $(2x - 1)^{1/2} + 5 = 6(2x - 1)^{-1/2}.$

25. $(x^2 - x)^2 - 8(x^2 - x) = -12.$

26. $(x^2 - 5x)^2 + (x^2 - 5x) = 12.$

27. $(2x^2 + x)^2 = 4(2x^2 + x) - 3.$

28. $(3x^2 + 2x)^2 + 5 = 6(3x^2 + 2x).$

29. $\left(\dfrac{x-1}{x}\right)^2 - \dfrac{x-1}{x} - 2 = 0.$

30. $\left(\dfrac{2x}{x-1}\right)^2 - 7\left(\dfrac{2x}{x-1}\right) = 30.$

31. $5\left(\dfrac{x+1}{x-7}\right)^2 + 3\left(\dfrac{x+1}{x-7}\right) = 2.$

32. $\dfrac{1}{(x+2)^2} + \dfrac{5}{x+2} + 4 = 0.$

33. $\dfrac{2x-3}{x} + \dfrac{7x}{2x-3} - 8 = 0.$

34. $4x^2 - 3x + \dfrac{5}{4x^2 - 3x} - 6 = 0.$

35. $\dfrac{x^2-3}{x-1} = 2\left(\dfrac{x-1}{x^2-3}\right) - 1.$

36. $\dfrac{x+1}{x-3} - \sqrt{\dfrac{x+1}{x-3}} - 2 = 0.$

79. Equations Containing Radicals. An equation in which the unknown appears under a radical sign is called a **radical, or irrational, equation.**

Illustrations:

$\sqrt{3x^2 - 1} = x$ is a radical equation of second order.

$\sqrt[3]{x - 1} = 4$ is a radical equation of third order.

The usual process for solving a radical equation employs the operation of raising both members of an equation to the same positive integral power. The new equation thus obtained has all the roots of the original equation. This is true because the results of taking the same positive integral power of two equal numbers are equal. The derived equation, however, may have roots in addition to those of the original equation. The extra roots are called extraneous roots. The existence of extraneous roots is illustrated in the examples worked out below.

We list the steps for solving second order radical equations.

1. Arrange the members of the equation so that a single radical constitutes one member.

2. Eliminate this radical by squaring both sides of the equation.

3. Repeat steps 1 and 2, if necessary, until an equation free of radicals is obtained.

4. Solve the resulting equation and check all its roots in the given equation. Reject the extraneous roots, if any.

Example 1. Solve $\sqrt{x + 4} - \sqrt{2x + 1} + 1 = 0.$

Solution.

Transpose:

$$\sqrt{x + 4} = \sqrt{2x + 1} - 1.$$

Square:

$$x + 4 = 2x + 1 - 2\sqrt{2x + 1} + 1.$$

Transpose:

$$2\sqrt{2x + 1} = x - 2.$$

Square:

$$4(2x + 1) = x^2 - 4x + 4.$$

Transpose:

$$x^2 - 12x = 0,$$

$$x(x - 12) = 0,$$

$$x = 0 \text{ and } 12.$$

Check in given equation:

For $x = 0$	For $x = 12$
$\sqrt{4} - \sqrt{1} + 1 \overset{?}{=} 0,$	$\sqrt{16} - \sqrt{25} + 1 \overset{?}{=} 0,$
$2 - 1 + 1 \neq 0,$	$4 - 5 + 1 = 0,$
$2 \neq 0.$	$0 = 0.$

Though zero is a root of the quadratic equation, it is not a root of the given radical equation. Hence 12 is the only root of the radical equation.

Example 2. Solve $\sqrt{3x + 4} - \sqrt{2x + 1} - \sqrt{x - 3} = 0.$

Solution.

Transpose:

$$\sqrt{3x + 4} = \sqrt{2x + 1} + \sqrt{x - 3}.$$

Square:

$$3x + 4 = 2x + 1 + 2\sqrt{(2x + 1)(x - 3)} + x - 3.$$

Transpose:

$$6 = 2\sqrt{(2x + 1)(x - 3)},$$

$$3 = \sqrt{(2x + 1)(x - 3)}.$$

Square:

$$9 = 2x^2 - 5x - 3,$$

$$2x^2 - 5x - 12 = 0,$$

$$(2x + 3)(x - 4) = 0,$$

$$x = 4 \text{ and } -\tfrac{3}{2}.$$

Check:

For $x = 4$

$$\sqrt{12 + 4} - \sqrt{8 + 1} - \sqrt{4 - 3} \overset{?}{=} 0,$$

$$4 - 3 - 1 = 0,$$

$$0 = 0.$$

For $x = -\frac{3}{2}$

$$\sqrt{-\tfrac{9}{2} + 4} - \sqrt{-3 + 1} - \sqrt{-\tfrac{3}{2} - 3} \overset{?}{=} 0,$$

$$\sqrt{-\tfrac{1}{2}} - \sqrt{-2} - \sqrt{-\tfrac{9}{2}} \overset{?}{=} 0,$$

$$\tfrac{1}{2}\sqrt{-2} - \sqrt{-2} - \tfrac{3}{2}\sqrt{-2} \overset{?}{=} 0,$$

$$-2\sqrt{-2} \neq 0.$$

We reject the extraneous root, $-\frac{3}{2}$. The given equation has only the root 4.

Exercise 54

Solve the following equations. Check your results.

1. $\sqrt{3x + 1} = 4$.
2. $\sqrt[3]{2x + 3} = 5$.
3. $\sqrt[4]{4x - 5} = 1$.
4. $\sqrt{x^2 + 5} = 3$.
5. $\sqrt{x^2 - 3x} = 2$.
6. $\sqrt[3]{x^2 - 7x} = 2$.
7. $\sqrt{2x + 1} = 0$.
8. $\sqrt{3x + 2} = -5$.
9. $\sqrt{6x - 8} = x$.
10. $\sqrt{1 - 4x} = \sqrt{3x}$.
11. $\sqrt{5x + 6} = x + 2$.
12. $\sqrt{4 - x} = x - 4$.
13. $\sqrt{2x - 5} = x - 4$.
14. $\sqrt{1 - x} = x + 5$.
15. $\sqrt{2 + x} = x + 2$.

16. $\sqrt{2x + 13} - \sqrt{x + 10} = 1$.
17. $\sqrt{x - 2} - \sqrt{x - 5} = 1$.
18. $\sqrt{2x + 11} - \sqrt{x + 2} = 2$.
19. $1 + \sqrt{x + 9} = \sqrt{2x + 11}$.
20. $\sqrt{3x - 5} + \sqrt{x - 2} = 3$.
21. $\sqrt{5 - 4x} + \sqrt{13 - 4x} = 4$.
22. $\sqrt{x} + \sqrt{3x + 4} = \sqrt{6x + 12}$.
23. $\sqrt{x + 6} = \sqrt{6x + 6} - \sqrt{x}$.
24. $3\sqrt{x - 5} + \sqrt{x - 9} = 2\sqrt{x}$.
25. $2\sqrt{2x} - \sqrt{3x + 3} = \sqrt{x - 1}$.

26. $\sqrt{1 - x} + \sqrt{7 - x} - \sqrt{12 - 6x} = 0$.
27. $\sqrt{2x - 3} - \sqrt{7x - 5} + \sqrt{x + 2} = 0$.
28. $\sqrt{7 - 2x} - \sqrt{3 - x} - \sqrt{4 - x} = 0$.
29. $\sqrt{x^2 - x - 2} - \sqrt{x^2 + x + 7} + 1 = 0$.
30. $\sqrt{x^2 - 3x + 4} - \sqrt{x^2 - x + 3} - 1 = 0$.
31. $\sqrt{4x^2 - x - 8} - \sqrt{4x^2 + 3x + 4} + 2 = 0$.

80. The Nature of the Roots. The roots of the general quadratic equation are x_1 and x_2, where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The radicand $b^2 - 4ac$ is called the **discriminant.** The discriminant, which can be had without solving the equation, yields information regarding the nature of the roots. Recalling that we are restricting a, b, and c to real numbers, we make the following observations:

1. *If* $b^2 - 4ac = 0$, *the roots are real and equal.*
2. *If* $b^2 - 4ac > 0$, *the roots are real and unequal.*
3. *If* $b^2 - 4ac < 0$, *the roots are imaginary and unequal.*

If we further restrict the coefficients a, b, and c to be rational numbers, the following statements can be made:

If $b^2 - 4ac$ *is a perfect square, the roots are rational.*
If $b^2 - 4ac$ *is not a perfect square, the roots are irrational.*

These conclusions can be readily verified. If the discriminant is zero, the roots are

$$x_1 = \frac{-b + 0}{2a} \quad \text{and} \quad x_2 = \frac{-b - 0}{2a}.$$

These quantities are obviously equal and are real because a and b are both real. If the discriminant is not zero, the roots, of course, are unequal. It is also clear that the roots are imaginary if and only if each contains the square root of a negative discriminant. With a, b, and c rational, the roots are rational if and only if the discriminant is a perfect square (the square of a rational number).

Illustrations:

Equation	*Discriminant*	*Nature of Roots*
$5x^2 - 21x + 18 = 0$	$(-21)^2 - 4(5)(18) = 81$	Real, unequal, rational
$2x^2 + 8x + 3 = 0$	$8^2 - 4(2)(3) = 40$	Real, unequal, irrational
$x^2 - 10x + 25 = 0$	$10^2 - 4(25) = 0$	Real, equal, rational
$3x^2 + 6x + 4 = 0$	$6^2 - 4(3)(4) = -12$	Imaginary and unequal

Example. Determine k so that the roots of $kx^2 + 4x + k - 3 = 0$ are equal.

Solution. The roots are equal if the discriminant is zero. Hence, substituting in $b^2 - 4ac = 0$, we obtain

$$4^2 - 4(k)(k-3) = 0,$$
$$16 - 4k^2 + 12k = 0,$$
$$k^2 - 3k - 4 = 0,$$
$$(k+1)(k-4) = 0,$$
$$k = -1 \text{ and } 4.$$

The roots of the given equation are equal when k has the value -1 or the value 4.

Exercise 55

Without solving, determine the nature of the roots:

1. $4x^2 - 12x + 9 = 0.$ **2.** $3x^2 + 5x - 2 = 0.$ **3.** $4x^2 + 7x + 3 = 0.$
4. $7x^2 - 4x + 3 = 0.$ **5.** $x^2 - x + 4 = 0.$ **6.** $x^2 + 4x + 5 = 0.$
7. $x^2 - 7x + 7 = 0.$ **8.** $2x^2 + 5x + 5 = 0.$ **9.** $x - \sqrt{3}\,x - 1 = 0.$

10. $x^2 - 2\sqrt{2}\,x + 2 = 0.$ **11.** $\sqrt{3}\,x^2 - x - 2 = 0.$

12. $x^2 + 5x - \sqrt{2} = 0.$ **13.** $9x^2 = 6x - 1.$ **14.** $5x^2 + 9x = 2.$
15. $x^2 = 8x - 16.$

Determine k so that each equation has equal roots:

16. $x^2 + 3x + k = 0.$ **17.** $kx^2 - 6x + 3 = 0.$ **18.** $2x^2 - kx + k = 0.$

19. $x^2 - (k+1)x + 4k - 3 = 0.$ **20.** $(k+2)x^2 + 12x + 2k + 5 = 0.$
21. $x^2 - (k+5)x + 4k + 5 = 0.$ **22.** $kx^2 + (3k-1)x + k - 3 = 0.$

81. The Sum and Product of the Roots. Indicating the roots of the general quadratic equation $ax^2 + bx + c = 0$ by x_1 and x_2, we find their sum to be

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}.$$

Their product is

$$x_1x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

We put these results in words:

The sum of the roots of a quadratic equation in standard form is equal to the negative of the coefficient of the first degree term divided by the coefficient of the second degree term.

The product of the roots of a quadratic equation in standard form is equal to the constant term divided by the coefficient of the second degree term.

These formulas enable one to find the sum and product of the roots at a glance. There are situations in which the sum or product of the roots is needed but not the individual roots. The formulas may also be used as a rapid check after solving a quadratic equation.

Illustrations:

Equations	*Sum of Roots*	*Product of Roots*
$4x^2 - 7x + 8 = 0$	$-\frac{b}{a} = \frac{7}{4}$	$\frac{c}{a} = \frac{8}{4} = 2$
$x^2 - x - 1 = 0$	1	-1
$6 - 3x - 2x^2 = 0$	$\frac{3}{-2} = -\frac{3}{2}$	$\frac{6}{-2} = -3$
$7x^2 - 42x + 1 = 0$	$\frac{42}{7} = 6$	$\frac{1}{7}$

If the coefficient of the second degree term of a quadratic equation is unity ($a = 1$), the sum of the roots is $-b$ and the product c. This fact may be used to form a quadratic equation whose roots are two given numbers.

Example 1. Form a quadratic equation whose roots are $-\frac{3}{2}$ and 5.

Solution. By making the coefficient of the second degree term unity, we may use the negative of the sum of the given roots as the coefficient of the first degree term and their product as the constant term. Since $5 - \frac{3}{2} = \frac{7}{2}$ and $5(-\frac{3}{2}) = -\frac{15}{2}$, we have the desired equation

$$x^2 - \tfrac{7}{2}x - \tfrac{15}{2} = 0 \quad \text{or} \quad 2x^2 - 7x - 15 = 0.$$

Alternatively, the equation could be formed by reversing the operation of solving a quadratic equation by factoring. Thus we may write the equation

$$(x + \tfrac{3}{2})(x - 5) = 0.$$

We see at once that the roots are $-\frac{3}{2}$ and 5. Multiplying the two factors, we have, as before, the equation $x^2 - \frac{7}{2}x - \frac{15}{2} = 0$.

Example 2. Determine the value of k so that one root of the equation $2x^2 - 5x + k = 0$ is twice the other.

Solution. Let x_1 and $2x_1$ stand for the roots. Then, adding the roots, we have

$$x_1 + 2x_1 = -\frac{b}{a} = \frac{5}{2} \quad \text{and} \quad x_1 = \frac{5}{6}.$$

The roots, therefore, are $x_1 = \frac{5}{6}$ and $2x_1 = \frac{5}{3}$. Now the product of the roots yields

$$\frac{c}{a} = \frac{k}{2} = \left(\frac{5}{6}\right)\left(\frac{5}{3}\right) = \frac{25}{18} \quad \text{and} \quad k = \frac{25}{9}.$$

Assigning k this value in the given equation, we obtain

$$2x^2 - 5x + \tfrac{25}{9} = 0 \quad \text{or} \quad 18x^2 - 45x + 25 = 0.$$

The roots of this equation are $\frac{5}{6}$ and $\frac{5}{3}$.

Exercise 56

Without solving, find the sum and the product of the roots of the following equations:

1. $4x^2 - 8x + 3 = 0$.
2. $3x^2 - 2x - 6 = 0$.
3. $x^2 + 6x + 5 = 0$.
4. $y^2 + 5y = 7$.
5. $2y^2 - 6y - 3 = 0$.
6. $3y^2 = 7y + 2$.
7. $4 - 7x - 4x^2 = 0$.
8. $x^2 + 9 = 0$.
9. $x^2 - 9x = 0$.
10. $6x^2 + 18x = 0$.
11. $5y^2 + 3 = 0$.
12. $5 - 6y - y^2 = 0$.
13. $7x - 6 + 2x^2 = 3$.
14. $8x^2 - x + 6 = 4x$.
15. $2y^2 - 7y + 3 = 7$.

Using the formulas for the sum and the product of the roots, find quadratic equations whose roots are as follows:

16. $2, -1$.
17. $3, 4$.
18. $-2, -6$.
19. $\frac{2}{3}, -\frac{4}{3}$.
20. $\frac{2}{5}, \frac{5}{2}$.
21. $-\frac{3}{2}, \frac{4}{5}$.
22. $\sqrt{2}, -\sqrt{2}$.
23. $2i, -2i$.
24. $\sqrt{5}, -\sqrt{5}$.
25. $2 + \sqrt{2}, 2 - \sqrt{2}$.
26. $1 - \sqrt{3}, 1 + \sqrt{3}$.
27. $3 - i, 3 + i$.
28. $\frac{2}{3} + \frac{1}{3}i, \frac{2}{3} - \frac{1}{3}i$.
29. $\frac{2}{5} + \frac{3}{5}\sqrt{2}, \frac{2}{5} - \frac{3}{5}\sqrt{2}$.
30. $6 - i\sqrt{5}, 6 + i\sqrt{5}$.

Determine the values of k so that each given condition is satisfied:

31. $4x^2 - kx + 7 = 0$, sum of roots 3.
32. $kx^2 + 6x + 32 = 0$, sum of roots -2.
33. $7x^2 - 3kx + 41 = 0$, sum of roots 0.
34. $3x^2 + 41x + k = 0$, product of roots 7.
35. $8x^2 + 5x - 6k = 0$, product of roots 0.
36. $x^2 + 9x + k = 0$, one root double the other.

37. $x^2 - kx + 18 = 0$, one root double the other.
38. $4x^2 - 3x + k = 0$, one root three times the other.
39. $x^2 - 4x + k = 0$, difference of roots 6.
40. $x^2 - 7x + k - 4 = 0$, difference of roots 5.
41. $3kx^2 - 5x + 4k - 26 = 0$, product of roots -3.
42. $4x^2 + 10x + k - 3 = 0$, one root is the reciprocal of the other.
43. $5x^2 - (k + 2)x + 7k - 6 = 0$, one root is 2.
44. $kx^2 + (3k - 4)x - 5 = 0$, one root is $\frac{1}{2}$.
45. $4x^2 + 5kx - x + 2k + 7 = 0$, one root is $-\frac{2}{3}$.

82. Graph of a Quadratic Function. The expression $ax^2 + bx + c$, which is the left member of the general quadratic equation, is called a quadratic function. We wish now to consider the graphs of functions of this kind. The graph of a quadratic function is called a **parabola.** We shall construct the graphs of some quadratic functions, employing the procedure described in Section 41.

Example 1. Draw the graph of the function $2x^2 - 6x + 2$.

Solution. We let y stand for the function and write

$$y = 2x^2 - 6x + 2.$$

We next assign various values to x, compute the corresponding values of y, and tabulate the results. By plotting the points (x, y) of the table and drawing a smooth curve through them, we obtain the graph in Figure 9.

x	y
−0.5	5.5
0	2
0.5	−0.5
1	−2
1.5	−2.5
2	−2
2.5	−0.5
3	2
3.5	5.5

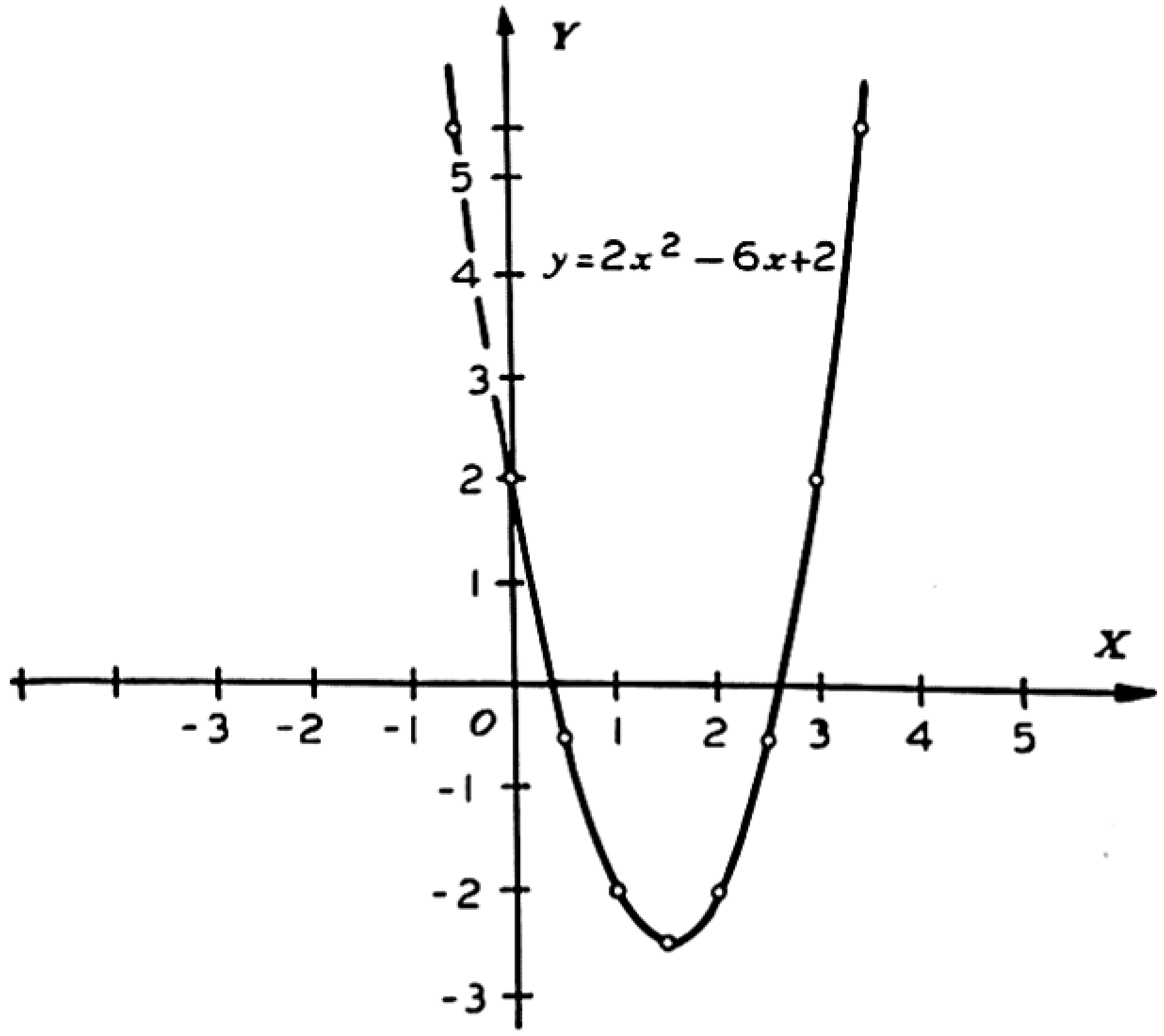


FIG. 9

Example 2. Draw the graph of the function $3 + 2x - x^2$.
Solution. Setting the function equal to y, we have

$$y = 3 + 2x - x^2.$$

We prepare a table of corresponding values of x and y, plot the points thus determined, and draw a curve through them (Fig. 10).

x	y
-2	-5
-1	0
0	3
1	4
2	3
3	0
4	-5

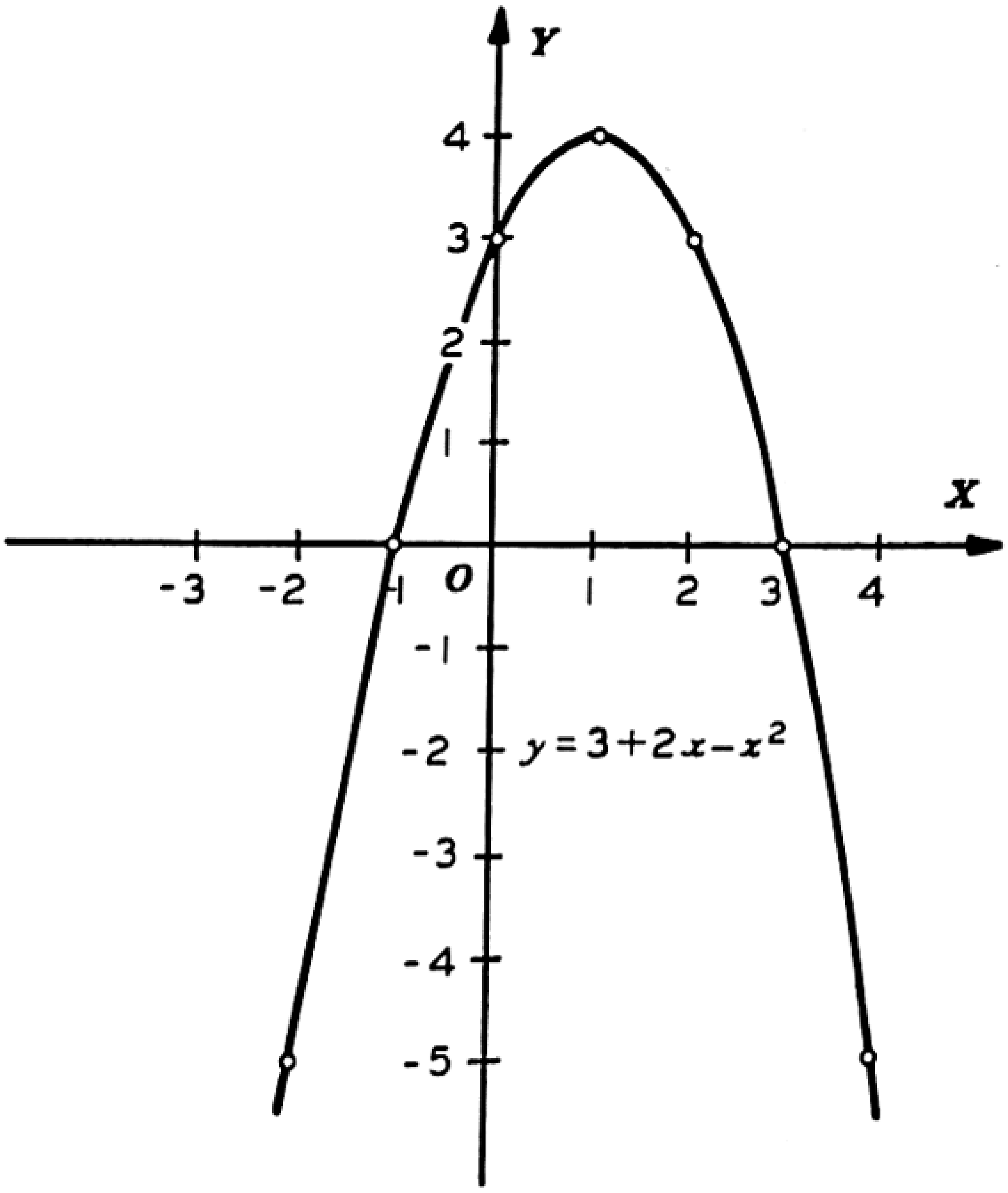


FIG. 10

The graph in Figure 9 is a parabola which opens upward, and that in Figure 10 opens downward. The direction in which a parabola opens is determined by the sign of the coefficient of the second degree term. It is proved in analytic geometry that the graph of $ax^2 + bx + c$ opens upward if a is positive and opens downward if a is negative.

Noticing further the graph (Fig. 9) of $y = 2x^2 - 6x + 2$, we see that it crosses the x-axis at two points. The abscissa of each of these points is a value of x for which the function $2x^2 - 6x + 2$ is equal to zero. Hence these values are zeros of the function and roots of the equation

$$2x^2 - 6x + 2 = 0.$$

Referring to the graph, we estimate the roots to be $x = 0.4$ and $x = 2.6$.

These are approximate values of the roots. The exact values, by the quadratic formula, are $\frac{1}{2}(3 + \sqrt{5})$ and $\frac{1}{2}(3 - \sqrt{5})$.

The graph of $y = 3 + 2x - x^2$ (Fig. 10) crosses the x-axis at $x = -1$ and $x = 3$. Hence these values are roots of the equation

$$3 + 2x - x^2 = 0.$$

The preceding examples illustrate a graphical method of solving a quadratic equation. Only the real roots, however, can be found by this method, and usually the results are approximations to the exact values.

It may be observed that the graph of $ax^2 + bx + c$ does not touch the x-axis if $b^2 - 4ac < 0$. The negative discriminant means that there is no real number which makes $ax^2 + bx + c$ equal to zero. On the other hand, the graph crosses the x-axis at two points if $b^2 - 4ac > 0$, and touches the x-axis at one point only if $b^2 - 4ac = 0$. Hence the discriminant reveals if the graph of a quadratic function crosses the x-axis at two points, touches the x-axis in only one point, or has no point on the x-axis. These three possibilities are illustrated by the graphs (Fig. 11) of the functions $y = x^2 - 2x - 1$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x + 3$, whose discriminants are respectively positive, zero, and negative.

$y =$
$x^2 - 2x - 1.$

x	y
-1	2
0	-1
1	-2
2	-1
3	2
4	7

$y =$
$x^2 - 2x + 1.$

x	y
-1	4
0	1
1	0
2	1
3	4
4	9

$y =$
$x^2 - 2x + 3.$

x	y
-1	6
0	3
1	2
2	3
3	6
4	11

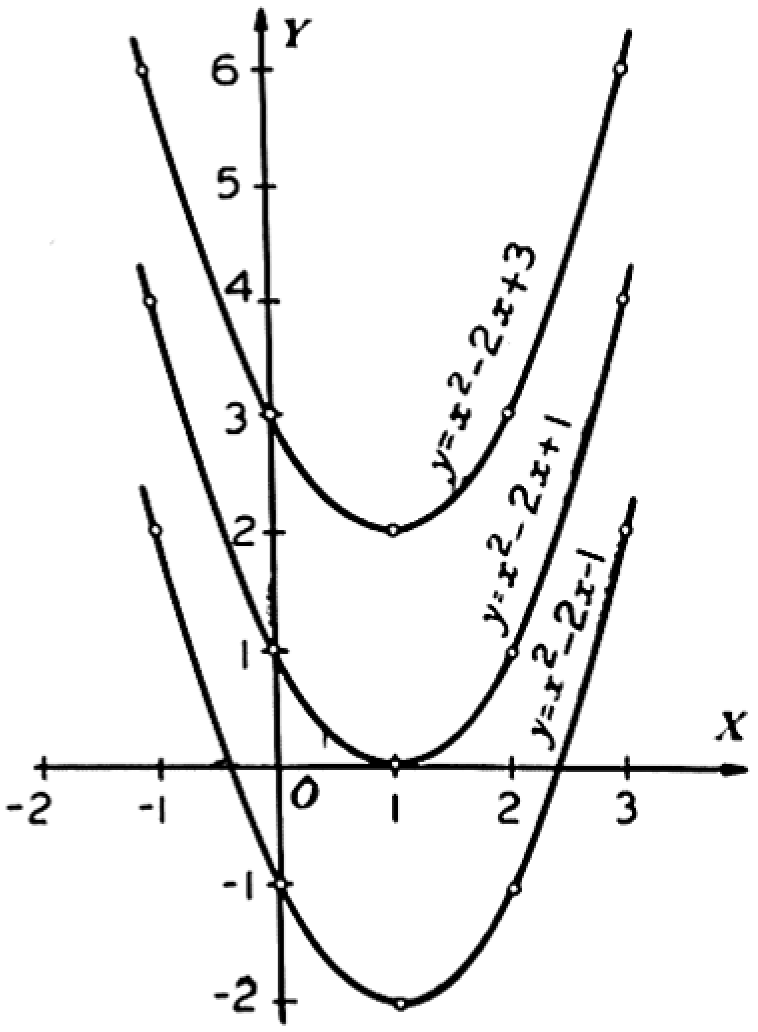


FIG. 11

83. Maximum and Minimum Values. The graph of $ax^2 + bx + c$ opens upward if $a > 0$, and the curve has a lowest point. The abscissa of the lowest point is the value of x for which the function has its least, or **minimum,** value. The ordinate of the point is the minimum value of the function. Similarly, the graph opens downward if $a < 0$, and the curve has a highest point. The abscissa of the highest point is the value of x for which the function has its greatest, or **maximum,** value. The coordinates of a minimum point or a maximum point can be obtained approximately from the graph. The exact coordinates can be determined directly from the function as is illustrated in the following examples.

Example 1. Determine the coordinates of the minimum point of the graph of the function $2x^2 - 6x + 2$.

Solution. This is the function whose graph is shown in Figure 9. However, we shall find the coordinates of the minimum point without the use of the graph. First we factor 2 from the x terms and have

$$2(x^2 - 3x) + 2.$$

The terms within the parentheses become a perfect square by the addition of $\frac{9}{4}$. Since these terms are multiplied by 2, we subtract $\frac{9}{2}$ in order that the value of the function shall be unchanged. Thus we get

$$2(x^2 - 3x + \tfrac{9}{4}) + 2 - \tfrac{9}{2},$$

or

$$2(x - \tfrac{3}{2})^2 - \tfrac{5}{2}.$$

The first term $2(x - \frac{3}{2})^2$ cannot be negative. It is equal to zero when $x = \frac{3}{2}$ and is positive for all other values of x. When $x = \frac{3}{2}$, the value of the function is $-\frac{5}{2}$. This obviously is the least value of the function. Hence the coordinates of the minimum point of the graph are $(\frac{3}{2}, -\frac{5}{2})$.

Example 2. Determine the value of x for which the function $3 + 2x - x^2$ has its maximum value.

Solution. We complete the square in the x terms by subtracting 1. It is necessary of course to add 1 to retain the original value of the function. Thus we get

$$\begin{aligned} 3 + 2x - x^2 &= 3 - (x^2 - 2x) \\ &= 3 + 1 - (x^2 - 2x + 1) \\ &= 4 - (x - 1)^2. \end{aligned}$$

It is evident that the function has the value 4 when $x = 1$. When x takes a value other than 1, the value of the function is 4 minus a positive quantity. Hence 4 is the maximum value of the given function. This result may also be obtained from the graph of the function (Fig. 10).

There are numerous practical problems whose solutions depend on minimum and maximum values of quadratic functions.

Example 3. What is the largest rectangular plot which can be fenced on three sides with 160 yards of fence?

Solution. We let x stand for the width, z for the length, and y for the area. Since one side is left unfenced, the length of the fence is $2x + z$. Hence we have

$$2x + z = 160 \quad \text{and} \quad z = 160 - 2x.$$

Then the area is given by

$$\begin{aligned} y &= xz \\ &= x(160 - 2x) \\ &= 160x - 2x^2 \\ &= -2(x^2 - 80x) \\ &= -2(x^2 - 80x + 1600) + 3200 \\ &= -2(x - 40)^2 + 3200. \end{aligned}$$

We see that the area is greatest when $x = 40$ yards. For this value of x the value of z is 80 yards. Hence the largest plot is obtained when the dimensions are 40 yards by 80 yards. The area is 3200 square yards.

Exercise 57

Construct the graphs of the functions given in problems **1–9**. Estimate, to one decimal place, the zeros of each function whose graph touches the x-axis.

1. $x^2 - 3x + 2.$ **2.** $x^2 - 4x + 4.$ **3.** $x^2 + 7x + 12.$
4. $1 - 2x - x^2.$ **5.** $3 + x - x^2.$ **6.** $5 + 3x - x^2.$
7. $\frac{1}{2}x^2 - x + 3.$ **8.** $x^2 + x + 5.$ **9.** $2x^2 + 2x + 1.$

Solve equations in problems **10–15** graphically. Estimate the roots to one decimal place.

10. $2x^2 - x - 3 = 0.$ **11.** $2x^2 - 3x - 7 = 0.$ **12.** $3x^2 + x - 1 = 0.$
13. $\frac{1}{4}x^2 + x - 2 = 0.$ **14.** $x^2 - 4x + 2 = 0.$ **15.** $x^2 - 5x + 1 = 0.$

Without drawing the graphs, determine how many points the graph of each function in problems **16–27** has on the x-axis. Also find the maximum or minimum value of each function.

16. $x^2 - 8x + 10.$ **17.** $1 - 6x - x^2.$ **18.** $x^2 + 4x + 3.$
19. $4 + 8x - x^2.$ **20.** $4x^2 + 4x + 1.$ **21.** $x^2 - 6x + 9.$
22. $2x^2 - 3x + 2.$ **23.** $-2x^2 + 2x - 5.$ **24.** $-3x^2 + 9x + 4.$
25. $3x^2 - 3x + 4.$ **26.** $9x^2 - 12x + 4.$ **27.** $-6x^2 + 4x + 3.$

28. What are the dimensions of the largest rectangular field which can be enclosed with 440 rods of fence?

29. Separate 18 into two parts such that the product of the parts is a maximum.

30. A rectangular pasture is to be fenced along four sides and divided into two equal parts by a fence parallel to one of the sides. Find the greatest area if the total length of the fence is 300 rods.

31. Separate the number 8 into two parts such that the sum of their squares is a minimum.

84. Stated Problems. There are many kinds of stated problems which can be solved by means of quadratic equations. Frequently only one root of the equation has meaning in the problem at hand. The root not satisfying the conditions of the problem must be rejected. It is well, therefore, to examine the roots to determine if one or both may be used.

Example 1. The square of a number exceeds the number by 56. Find the number.

Solution. Let x = the number. Then

$$x^2 - x = 56 \quad \text{or} \quad x^2 - x - 56 = 0.$$

$$(x - 8)(x + 7) = 0, \quad x = -7 \text{ and } 8.$$

Check:

$$8^2 - 8 = 56, \qquad (-7)^2 - (-7) = 56,$$

$$64 - 8 = 56, \qquad 49 + 7 = 56,$$

$$56 = 56. \qquad 56 = 56.$$

The problem has two solutions, -7 and 8.

Example 2. A rectangular garden, 80 feet long and 40 feet wide, has a walk of uniform width around it. Find the width of the walk if its area is 1300 square feet.

Solution. Let x stand for the width of the walk. Then the outside dimensions of the rectangle formed by the garden and the walk are $2x + 80$ and $2x + 40$. Hence the combined area of the garden and the walk is $(2x + 80)(2x + 40)$. Since the area of the walk is the combined area minus the area of the garden, we have the equation

$$(2x + 80)(2x + 40) - (80)(40) = 1300.$$

Then

$$4x^2 + 240x + 3200 - 3200 = 1300,$$

$$4x^2 + 240x - 1300 = 0,$$

$$x^2 + 60x - 325 = 0,$$

$$(x + 65)(x - 5) = 0,$$

$$x = -65 \text{ and } 5.$$

The width of the walk is positive; hence we select 5 feet for the width, which satisfies the conditions of the problem.

Example 3. A car which travels 10 miles per hour faster than a truck goes 360 miles in 3 hours less time than required by the truck to travel the same distance. Find the rate of each.

Solution. Let x = rate of the truck, and $x + 10$ = rate of car. Then

$$\frac{360}{x} = \text{time required by truck,}$$

and

$$\frac{360}{x + 10} = \text{time required by car.}$$

Since the difference of the times is 3, we write the equation

$$\frac{360}{x} - \frac{360}{x + 10} = 3;$$

whence, clearing of fractions,

$$\begin{aligned} 360(x + 10) - 360x &= 3x(x + 10), \\ 360x + 3600 - 360x &= 3x^2 + 30x, \\ 3600 &= 3x^2 + 30x, \\ x^2 + 10x - 1200 &= 0, \\ (x - 30)(x + 40) &= 0, \\ x &= 30 \text{ and } -40. \end{aligned}$$

Rejecting the negative quantity, we have $x = 30$, rate traveled by the truck; and $x + 10 = 40$, rate traveled by the car.

Exercise 58

1. The product of two consecutive odd integers is 143. Find the integers.

2. The sum of the squares of two consecutive integers is 85. Find the numbers.

3. The square of a number exceeds four times the number by 140. Find the number.

4. Two numbers differ by 1; their cubes differ by 91. What are the numbers?

5. The area of a rectangle is 88 square inches. What are the dimensions if the length is 3 inches more than the width?

6. The volumes of two cubes differ by 98 cubic inches; their edges differ by 2 inches. Find the edge of each cube.

7. The diagonal of a rectangle is 8 inches more than the length, and the length is 1 inch more than the width. Find the dimensions of the rectangle.

8. The dimensions of a page are 9 inches by 12 inches. The area of the four margins, which are all of the same width, is 38 square inches. Find the width of the margins.

9. A page, with length 3 inches more than its width, has 80 square inches of printed area. The margins at the top and bottom are each $1\frac{1}{2}$ inches wide, and the side margins are each 1 inch wide. Find the dimensions of the page.

10. A rectangular-shaped gutter is made by bending up the sides of a strip of sheet metal. If the width of the strip is 18 inches and the carrying area is $40\frac{1}{2}$ square inches, find the width and depth of the gutter.

11. Two boys are to mow a rectangular grass area 90 feet by 120 feet. One of the boys starts on the edge and mows around the perimeter. How wide a strip should he mow from each side in order to do half the work?

12. A skating rink 50 feet by 75 feet is to be enlarged into a rectangle of twice the area by adding strips of equal width on one end and a side. Find the width of the strips.

13. The length of a rectangular piece of tin is 6 inches more than its width. A 2-inch square is cut from each corner, and the edges are bent up to make a box of depth 2 inches and volume 112 cubic inches. Find the dimensions of the piece of tin.

14. A car travels 15 miles per hour faster than a truck and goes a distance of 220 miles in $1\frac{1}{2}$ hours less time than required by the truck to travel the same distance. Find the speed of each vehicle.

15. A man drove a distance of 120 miles at a certain speed and returned at a 15 miles per hour greater speed. Find the speeds if the total time of driving was $4\frac{2}{3}$ hours.

16. A ship leaves port and sails east at 20 miles per hour. A second ship leaves the port 1 hour later and sails south at 10 miles per hour. When will the ships be 40 miles apart?

17. A man is walking north at 3 miles per hour. A second man is walking east at 4 miles per hour. The first man reaches the intersection point of their paths 2 hours before the second man reaches it. At what times are the men 5 miles apart?

18. An airplane takes off from a field and travels east at 150 miles per hour. One hour later a second airplane leaves from the same field and travels south at 200 miles per hour. How long will it be until the airplanes are 200 miles apart?

19. An airplane travels 175 miles with a 25-mile-per-hour wind in 24 minutes less time than it takes to return against the wind. What is the still air speed of the plane?

20. A railroad and highway intersect at right angles. A train is 10 miles from the crossing at the same time a car is 6 miles from the crossing. If each is approaching the crossing and each is traveling at $\frac{1}{2}$ mile per minute, in how many minutes will they be 4 miles apart?

21. A man can do a job in 9 days less time than is required by a boy. Working together, they can do the job in 20 days. How long would it take each alone to do the work?

22. A man bought some pigs for \$480. Had the price been \$1 less per pig, he would have received 16 more pigs for the \$480. How many pigs did he buy?

23. A group of children rented a skating rink for \$20. Fifteen of the group did not attend, and consequently each child present was required to increase his contribution by 30 cents. How many were in the original group?

24. One pipe fills a tank in 4 hours less time than it takes a second pipe. Both pipes fill the tank in $2\frac{2}{3}$ hours. Find the time required for each pipe alone to fill the tank.

25. A can do a piece of work in 7 days less time than it takes B. Together they can do the work in 12 days. Find the time required for each alone to do the work.

26. If an object is thrown vertically downward with a velocity of v_0, the distance s which the object has fallen at the end of t seconds is given by the formula

$$s = v_0t + \tfrac{1}{2}gt^2$$

where s is in feet, v_0 in feet per second, and g is approximately 32.

From a balloon a marble is thrown downward with a velocity of 64 feet per second. Find the time required for the marble to fall 192 feet.

27. If a body is projected vertically upward, the formula for its distance from the starting point is

$$s = v_0t - \tfrac{1}{2}gt^2,$$

where v_0 is the initial velocity and $g = 32$.

A ball is thrown vertically upward with a starting velocity of 80 feet per second. In how many seconds will the ball be (a) 100 feet high; (b) 96 feet high? Explain the two answers for part (b).

28. A bullet is shot vertically upward at a balloon. If the bullet is shot at 1100 feet per second and the balloon is 9400 feet high, find the time required for the bullet to reach the balloon. (See problem **27.**)

Chapter Eleven

SYSTEMS INVOLVING QUADRATIC EQUATIONS

85. Quadratic Equations in Two Unknowns. An equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

where A, B, C, D, E, and F are any constants, except that A, B, and C cannot all be zero, is called a **general quadratic equation** in the two unknowns x and y. A pair of numbers for the variables x and y which satisfy the equation is called a solution of the equation. Two such equations taken together constitute a system of quadratic equations; and a pair of values for x and y which satisfies both equations is called a solution of the system.

We shall consider in this chapter the problem of solving two equations in two unknowns where (a) one equation is linear and the other quadratic, and (b) both equations are of second degree, or quadratic. Before attacking this problem, however, we shall construct the graphs of certain special cases of equation (1).

86. Graph of a Quadratic Equation. In Section 82 we constructed the graphs, called parabolas, of quadratic functions of x. That is, we constructed the graphs of equations of the form

$$y = ax^2 + bx + c.$$

An equation of this form may be obtained from equation (1) by assigning B and C the value zero and solving for y. Similarly, the equation

$$x = ay^2 + by + c$$

is a special case of equation (1). The graph of this equation is a parabola which opens to the right if $a > 0$ and to the left if $a < 0$ (see Fig. 16).

To construct the graph of an equation of the form (1) where all terms are present is a tedious task. Hence, as in the case of the parabola, we shall use equations which lack certain terms. The process comprises the following steps:

1. Solve the equation for y in terms of x. (If it is easier to solve for x, interchange x and y in these directions.)

2. Assign values to x and compute the corresponding values of y.

3. Plot the points thus determined and draw a smooth curve through them.

A. *The Circle.* The graph of an equation of the form

$$x^2 + y^2 = r^2$$

is a circle of radius r with the center at the origin. That the graph is a circle may be seen by referring to Figure 12. According to the Pythagorean theorem the coordinates (x, y) of points on the circle satisfy the equation $x^2 + y^2 = r^2$.

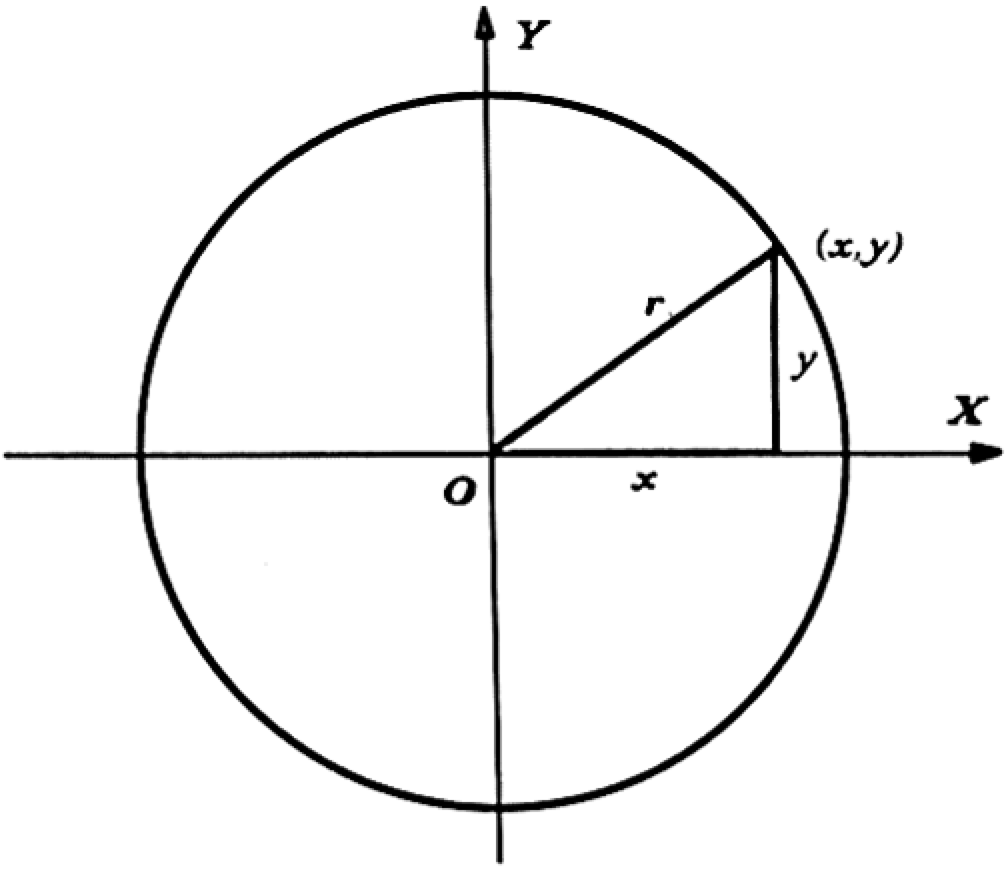


FIG. 12

Illustration: The graph of $x^2 + y^2 = 36$ is a circle of radius 6 with the center at the origin. It is easier to draw the circle with compasses than to use the point plotting process.

B. *The Ellipse.* The graph of an equation of the form

$$ax^2 + by^2 = c,$$

where a, b, and c are positive constants is called an **ellipse.**

Example 1. Construct the graph of the equation $4x^2 + 9y^2 = 36$.
Solution. We first solve the equation for y and have

$$y = \pm\tfrac{2}{3}\sqrt{9 - x^2}.$$

From this equation we see that if $x^2 > 9$, the radicand is negative and hence y is imaginary. This means that, for the purpose of graphing, x may take only the values from -3 to 3, inclusive. We assign several values to x in this range and find the corresponding values of y. The pairs of values for x and y determine points through which the curve of Figure 13 is drawn.

x	y
-3	0
-2.5	± 1.1
-2	± 1.5
-1	± 1.9
0	± 2
1	± 1.9
2	± 1.5
2.5	± 1.1
3	0

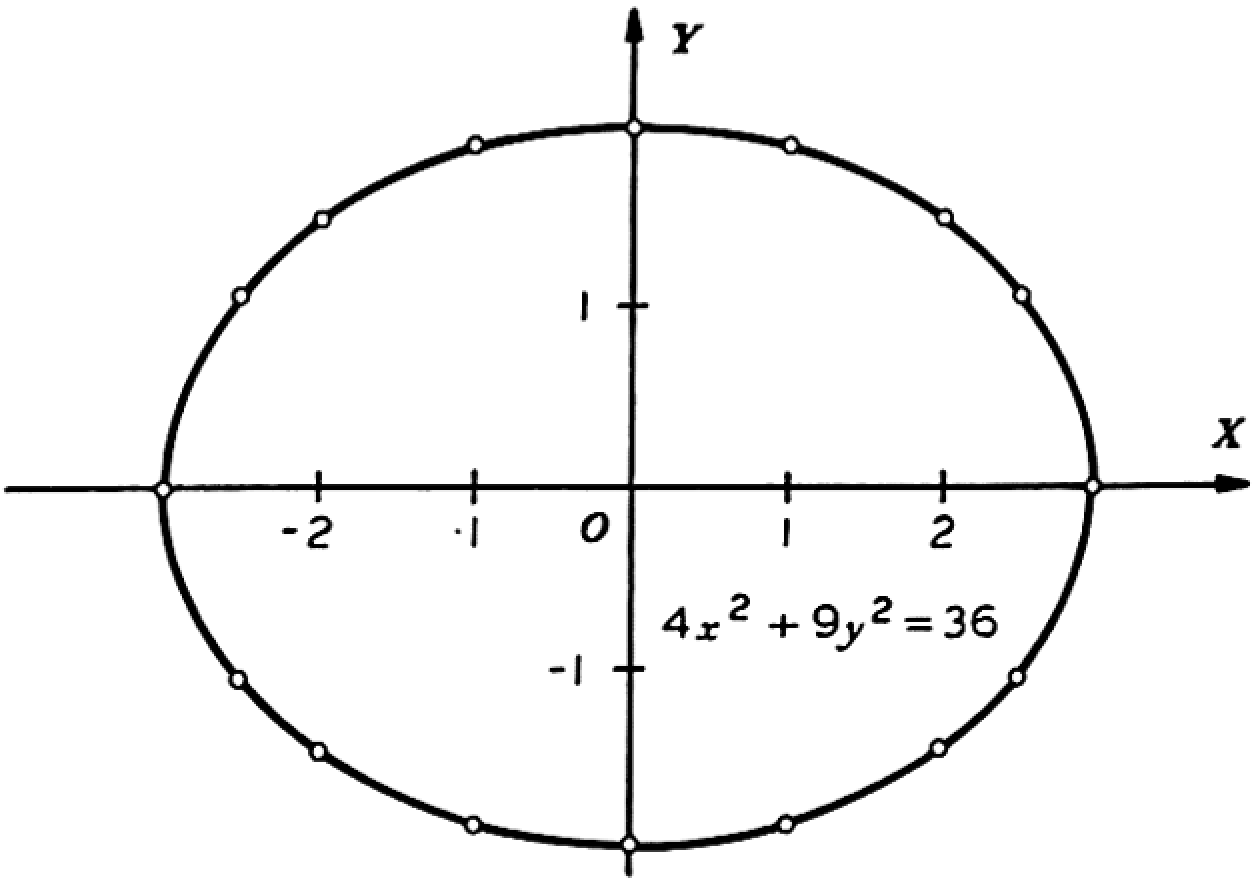


FIG. 13

C. *The Hyperbola.* The graph of an equation of the form

$$ax^2 - by^2 = c \quad \text{or} \quad ay^2 - bx^2 = c,$$

where a, b, and c are positive constants, is called a **hyperbola.**

Example 2. Construct the graph of the equation $16x^2 - 9y^2 = 144$.

x	y
-7	± 8.4
-6	± 6.9
-5	± 5.3
-4	± 3.5
-3.5	± 2.4
-3	± 0
3.5	± 2.4
4	± 3.5
5	± 5.3
6	± 6.9
7	± 8.4

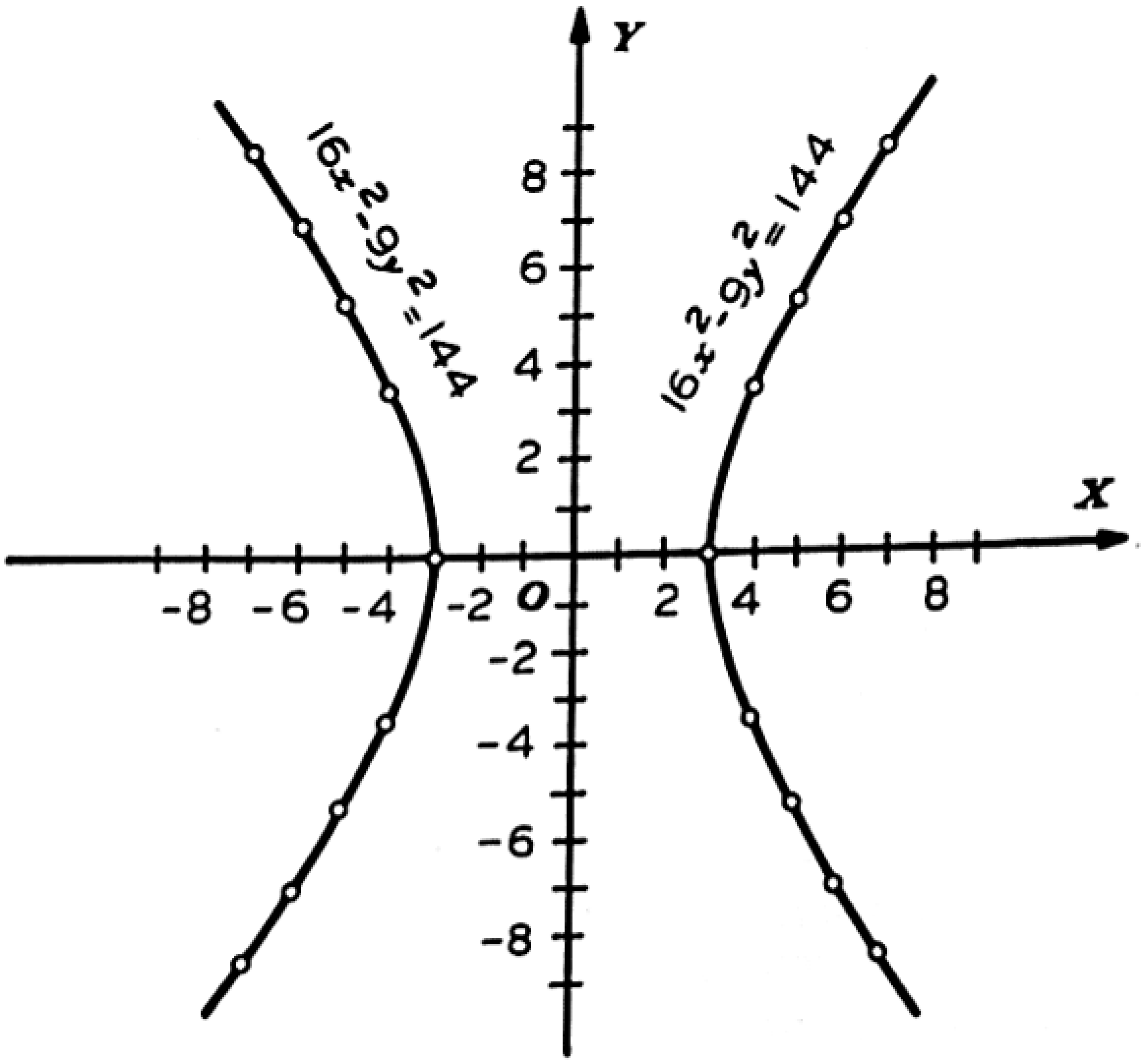


FIG. 14

Solution. Solving for y, we get

$$y = \pm\tfrac{4}{3}\sqrt{x^2 - 9}.$$

We notice that y is imaginary for values of x between -3 and 3. Hence the graph consists of two parts, one starting at (3, 0) and extending to the right, the other at $(-3, 0)$ and extending to the left. The graph, drawn by use of the pairs of values in the table, is in Figure 14.

The graph of $ay^2 - bx^2 = c$, where a, b, and c are positive constants, has one part which opens upward and the other part downward. This is illustrated in Figure 17. Each part of a hyperbola is called a **branch.**

Example 3. Draw the graph of the equation $xy = 4$.

Solution. In this equation x may be given any value except zero. If x were zero, there is no value for y which would make the product xy equal to 4. Similarly, y may have any value except zero. The graph is a hyperbola and comprises the two branches drawn in Figure 15.

x	y
-8	-0.5
-4	-1
-2	-2
-1	-4
-0.5	-8
0.5	8
1	4
2	2
4	1
8	0.5

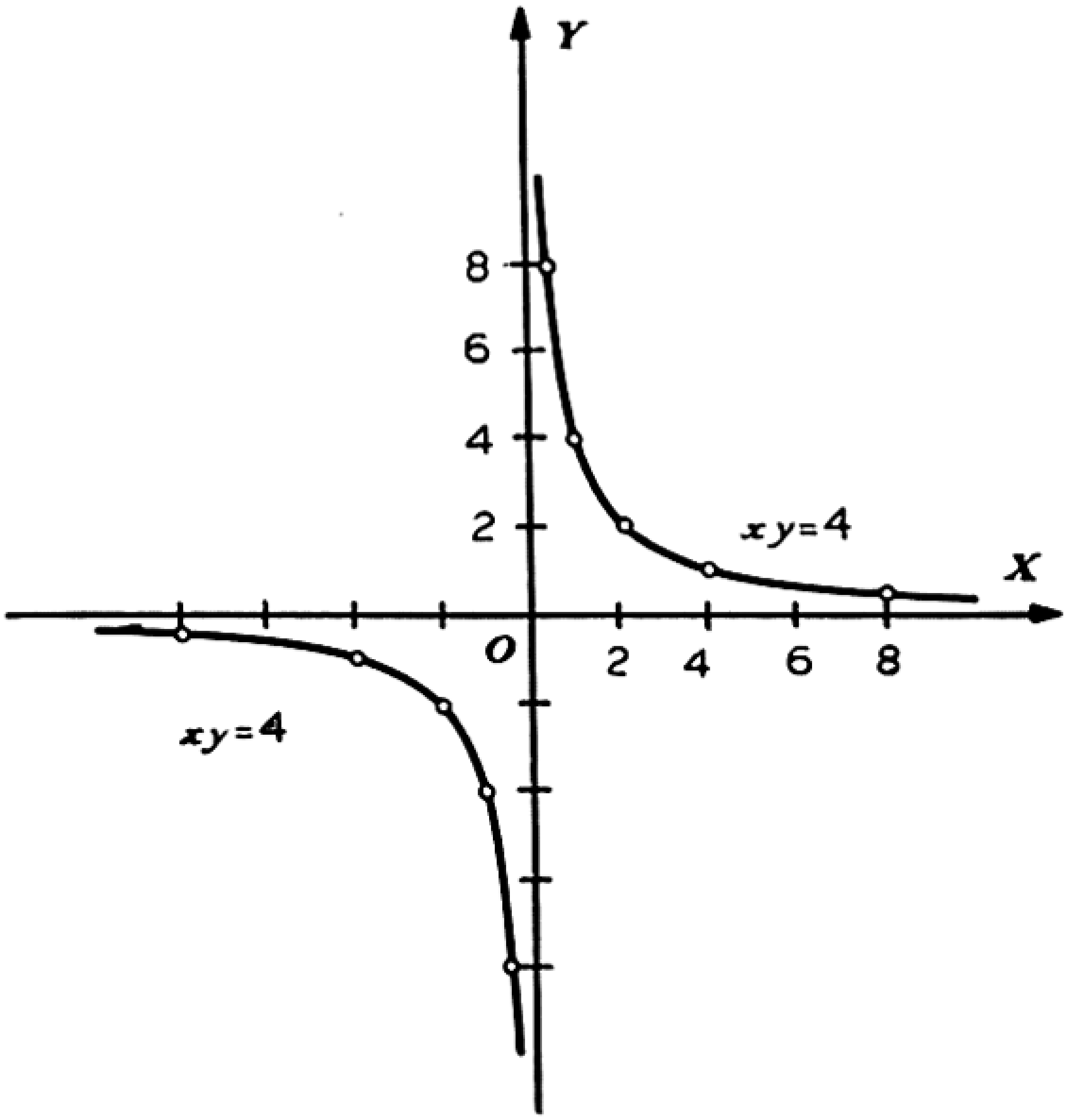


FIG. 15

87. Solution by Graphing. In Chapter Six we solved systems of linear equations by graphing each equation of a system and reading the coordinates of the point of intersection. The same procedure may be applied to second degree equations in two unknowns. However, the graphs will not be restricted to straight lines, and there may be more than one point of intersection. Only real solutions will be obtained graphically; and these, as a rule, will be approximations.

Example 1. Solve graphically the system

$$y^2 - 4x - 8 = 0$$

$$3x - 2y + 1 = 0.$$

Solution. We make a table of corresponding values for each equation. A table for the quadratic equation may be conveniently formed by solving for x and then assigning values to y.

$$x = \frac{y^2 - 8}{4}$$

x	y
-2	0
-1.75	± 1
-1	± 2
0.25	± 3
2	± 4
4.25	± 5

$$3x - 2y + 1 = 0$$

x	y
-1	-1
3	5

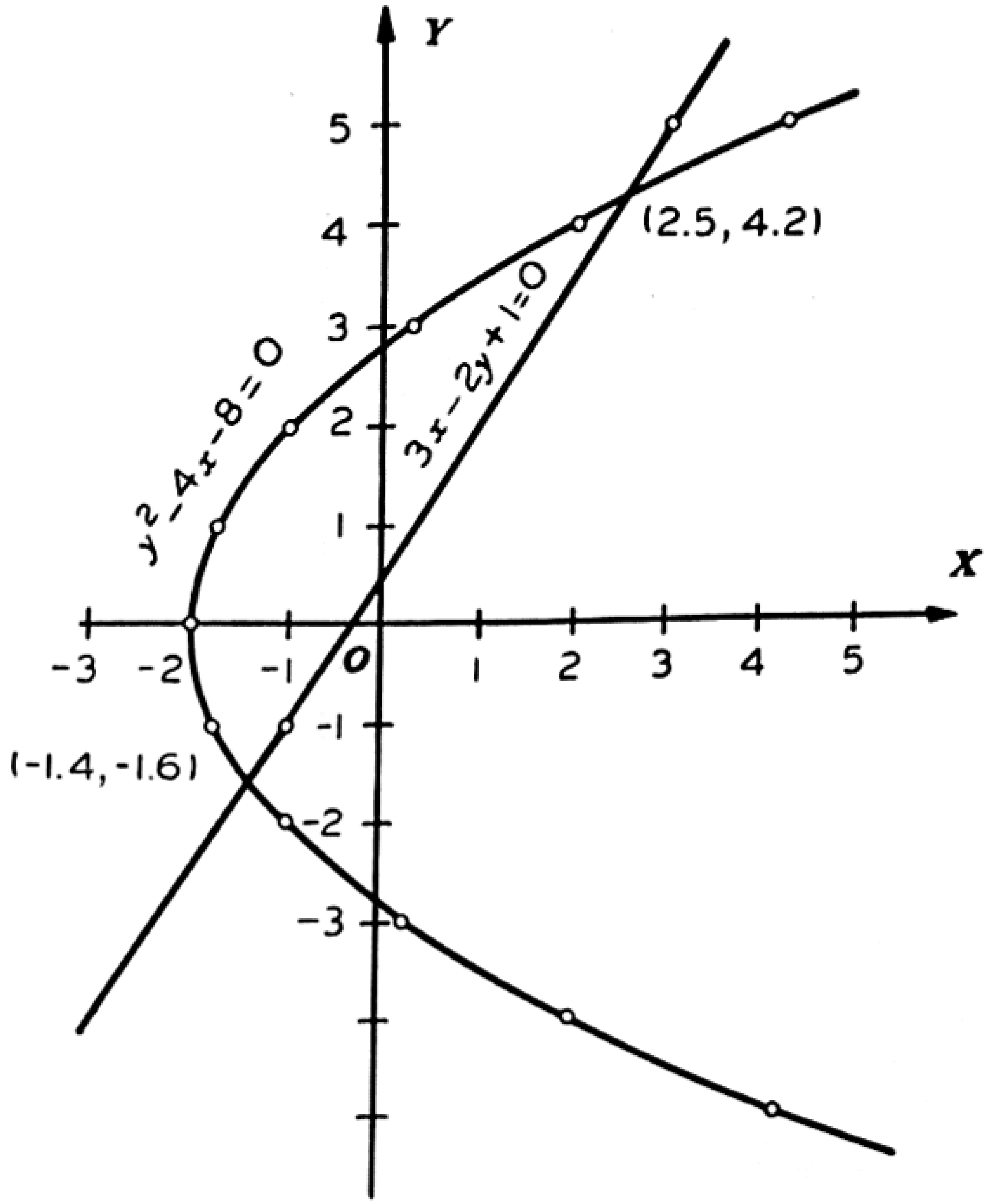


FIG. 16

The graphs of the two equations are shown in Figure 16. There are two intersection points, whose coordinates we estimate to be $(-1.4, -1.6)$ and $(2.5, 4.2)$. These value pairs may be written as $x = -1.4$, $y = -1.6$ and $x = 2.5$, $y = 4.2$. The results are good approximations to the solutions, as may be verified by substituting in the given equations.

Example 2. Solve graphically the system

$$4y^2 + 9x^2 = 36$$

$$9y^2 - 4x^2 = 36.$$

Solution. We solve each equation for y and then prepare tables of value pairs.

$y = \pm\frac{3}{2}\sqrt{4 - x^2}$

x	y
-2	0
-1.5	± 2.0
-1	± 2.6
0	± 3
1	± 2.6
1.5	± 2.0
2	0

$y = \pm\frac{2}{3}\sqrt{x^2 + 9}$

x	y
-3	± 2.8
-2	± 2.4
-1	± 2.1
0	± 2
1	± 2.1
2	± 2.4
3	± 2.8

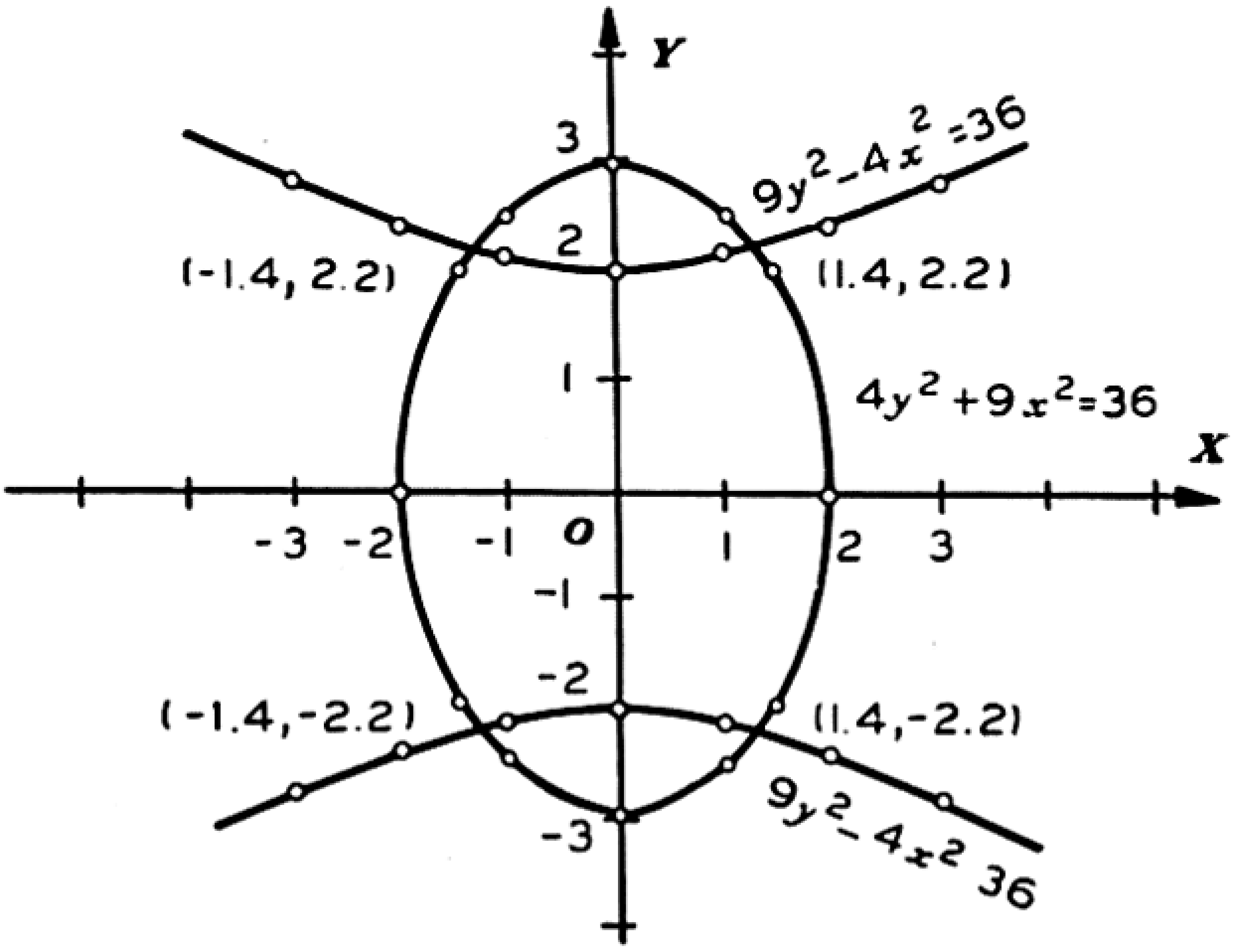


FIG. 17

The graphs (Fig. 17) intersect in four points whose coordinates are approximately $(-1.4, -2.2)$, $(-1.4, 2.2)$, $(1.4, -2.2)$, and $(1.4, 2.2)$.

Exercise 59

Construct and give the name of the graph of each of the following equations. (See Table I, page 301, to find square roots of numbers.)

1. $x^2 + y^2 = 49.$
2. $2x^2 + 2y^2 = 5.$
3. $x^2 + y^2 = 10.$
4. $x^2 + 4y^2 = 16.$
5. $4x^2 + 9y^2 = 36.$
6. $4x^2 + y^2 = 64.$
7. $4x^2 + 3y^2 = 12.$
8. $6x^2 + 5y^2 = 30.$
9. $x^2 + 25y^2 = 25.$
10. $x^2 - y^2 = 25.$
11. $9x^2 - 4y^2 = 36.$
12. $3y^2 - 2x^2 = 6.$
13. $2y^2 - 2x^2 = 5.$
14. $y^2 - 25x^2 = 100.$
15. $x^2 - 36y^2 = 36.$
16. $y^2 - 4x = 0.$
17. $y^2 - 4y - x = 0.$
18. $x + 4y - y^2 = 0.$
19. $xy - 4 = 0.$
20. $xy + 10 = 0.$
21. $3xy - 11 = 0.$

Solve the following systems graphically. Estimate the results to one decimal place.

22. $x^2 - 9y = 0$
$x - y = 0.$

23. $y^2 - 6x = 0$
$x + y = 1.$

24. $x^2 - y + 2 = 0$
$x - 2y + 5 = 0.$

25. $xy - 12 = 0$
$x + 2y = 2.$

26. $x^2 + y^2 = 25$
$7x + y = 4.$

27. $2x^2 + y^2 = 9$
$x - y - 1 = 0.$

28. $xy + 16 = 0$
$x^2 + y^2 = 36.$

29. $y^2 - x = 0$
$x^2 - 5y = 6.$

30. $2x^2 - 3y^2 = 8$
$3x^2 + 2y^2 = 8.$

31. $xy - 3 = 0$
$x^2 - y = 4.$

32. $2x^2 + 3y^2 = 12$
$x^2 - y = 3.$

33. $6y^2 - 9x^2 = 54$
$x^2 + y = 4.$

34. $4x^2 + y^2 = 16$
$x^2 + 4y^2 = 16.$

35. $9x^2 - 16y^2 = 144$
$4x^2 - 9y^2 = -36.$

36. $3x^2 + 4y^2 = 20.$
$y^2 - 9x = -9.$

88. Solution by Algebraic Methods. Any two independent quadratic equations in two unknowns can be solved by algebraic methods. These methods yield both the real and imaginary solutions exactly. In contrast, the graphical method does not give the imaginary solutions and usually gives the real solutions only approximately. However, a general algebraic method of solution would lead to a fourth degree equation. Though such an equation can be solved by algebraic methods to obtain its exact roots, we shall consider that problem as being too difficult at the present stage. But certain kinds of quadratic systems can be solved by special methods involving only processes which have been considered. A few of these systems will be discussed and illustrated by examples.

89. A System Without an *xy* and First Degree Terms. If both equations are of the form

$$Ax^2 + By^2 + F = 0,$$

we may employ one of the methods for solving two linear equations in two unknowns (Sec. 45). If we treat the equations as linear in x^2 and y^2, values for the squares of the unknowns can be found. Then the unknowns x and y are obtained by taking square roots. It is necessary, of course, to pair properly the values of x and y.

Example 1. Solve algebraically the system

$$x^2 + y^2 = 25$$

$$2x^2 - 3y^2 = 5.$$

Solution. Multiplying the members of the first equation by 2, and rewriting the second equation, we obtain

$$2x^2 + 2y^2 = 50$$

$$2x^2 - 3y^2 = 5.$$

By subtracting, we have

$$5y^2 = 45,$$

and

$$y = \pm 3.$$

Substituting either $y = 3$ or $y = -3$ in $x^2 + y^2 = 25$ gives

$$x^2 + 9 = 25,$$

$$x^2 = 16,$$

$$x = \pm 4.$$

We write the solution in pairs, and have

$x = 4,\ y = 3;\quad x = 4,\ y = -3;\quad x = -4,\ y = 3;\quad x = -4,\ y = -3.$

These solutions may also be conveniently paired as

$$(4, 3);\quad (4, -3);\quad (-4, 3);\quad (-4, -3);$$

and it can be shown by substitution that they satisfy the original equations. These are the coordinates of the points at which the two graphs would intersect.

Example 2. Solve the system

$$4x^2 + 3y^2 = 24$$

$$3x^2 - 2y^2 = 35.$$

Solution. Multiply the members of the first equation by 2 and those of the second by 3 to get

$$\begin{aligned} 8x^2 + 6y^2 &= 48 \\ 9x^2 - 6y^2 &= 105 \\ \hline 17x^2 &= 153, \\ x^2 &= 9, \\ x &= \pm 3. \end{aligned}$$

Substitute in $4x^2 + 3y^2 = 24$, and get

$$\begin{aligned} 36 + 3y^2 &= 24, \\ 3y^2 &= -12, \\ y^2 &= -4, \\ y &= \pm 2i. \end{aligned}$$

Pairing the values for x and y, we have $(3, 2i)$; $(3, -2i)$; $(-3, 2i)$; $(-3, -2i)$. The solutions being imaginary means that the graphs of the two equations have no point in common. These value pairs, however, satisfy each of the given equations.

Exercise 60

Solve the following systems of equations algebraically and check by substitution:

1. $x^2 + y^2 = 25$, $x^2 - y^2 = 7$.

2. $x^2 + y^2 = 29$, $4x^2 + y^2 = 41$.

3. $x^2 - y^2 = 11$, $3x^2 - 2y^2 = 58$.

4. $2x^2 + 3y^2 = 23$, $3x^2 - 2y^2 = 15$.

5. $4x^2 - 3y^2 = 29$, $x^2 + 7y^2 = 46$.

6. $5x^2 + 2y^2 = 37$, $3x^2 + 3y^2 = 15$.

7. $6x^2 - 11y^2 = 38$, $5x^2 - 9y^2 = 31$.

8. $7y^2 - 5x^2 = 17$, $4y^2 - 3x^2 = 12$.

9. $7x^2 + 2y^2 = 45$, $5x^2 + 3y^2 = 18$.

10. $8x^2 + 3y^2 = 2$, $5x^2 - 4y^2 = 13$.

11. $5y^2 - 6x^2 = 43$, $4y^2 - 5x^2 = 35$.

12. $4x^2 + 5y^2 = 21$, $3x^2 + 4y^2 = 16$.

90. A System with a Linear and a Quadratic Equation. If the system has a linear and a quadratic equation, the solution can be readily obtained. The following directions furnish a simple, workable plan:

1. Solve the linear equation for one unknown in terms of the other.

2. Substitute the result in the second degree equation. This gives a quadratic equation in one unknown.

3. Solve the quadratic equation to find the values of the unknown involved.

4. Substitute the values thus obtained in the linear equation of step 1 to find the corresponding values of the other unknown. Each pair of corresponding values is a solution of the given system.

Example 1. Solve the system

$$\begin{aligned} x^2 - 3xy - 2y^2 &= 2 \\ x - y &= 2. \end{aligned}$$

Solution. From the linear equation, $x = y + 2$. Substituting in the quadratic equation, we have

$$\begin{aligned} (y + 2)^2 - 3(y + 2)y - 2y^2 &= 2, \\ y^2 + 4y + 4 - 3y^2 - 6y - 2y^2 &= 2, \\ -4y^2 - 2y + 2 &= 0, \\ 2y^2 + y - 1 &= 0, \\ (2y - 1)(y + 1) &= 0, \\ y &= -1 \text{ and } \tfrac{1}{2}. \end{aligned}$$

Then

$$x = y + 2 = 1 \text{ and } \tfrac{5}{2}.$$

The value pairs $x = 1$, $y = -1$ and $x = \frac{5}{2}$, $y = \frac{1}{2}$ can be shown to satisfy each of the original equations. Hence, they are the solutions. The graphs of the two curves would intersect at the points $(1, -1)$ and $(\frac{5}{2}, \frac{1}{2})$.

Example 2. Solve the system

$$\begin{aligned} 4x^2 - 9y^2 - 8x - 36y - 68 &= 0 \\ y &= x - 2. \end{aligned}$$

Solution. Substitute $x - 2$ for y in the quadratic equation and get

$$4x^2 - 9(x - 2)^2 - 8x - 36(x - 2) - 68 = 0.$$

Whence,

$$5x^2 + 8x + 32 = 0,$$

$$x = \frac{-8 \pm \sqrt{64 - 640}}{10} = \frac{-8 \pm 24\sqrt{-1}}{10} = \frac{-4 \pm 12i}{5}.$$

$$y = x - 2 = \frac{-4 \pm 12i}{5} - 2 = \frac{-14 \pm 12i}{5}.$$

Pairing the values for x and y, we have

$$\left(\frac{-4 + 12i}{5}, \frac{-14 + 12i}{5}\right) \text{ and } \left(\frac{-4 - 12i}{5}, \frac{-14 - 12i}{5}\right).$$

The graphs of the two equations would not intersect since the solutions are imaginary.

Exercise 61

Solve each of the following systems algebraically:

1. $x^2 - y = 5$
$2x + y = 3.$

2. $y^2 + 2x = 7$
$x - 2y = -7.$

3. $2y^2 + 3x = 5$
$x - 3y = 4.$

4. $x^2 + y^2 = 10$
$x + y = 4.$

5. $x^2 + y^2 = 13$
$x + y = 1.$

6. $x^2 + y^2 = 20$
$3x - y = 2.$

7. $2x^2 + y^2 = 27$
$2x + y = 3.$

8. $3x^2 + 2y^2 = 25$
$x + y = 5.$

9. $4x^2 - y^2 = -5$
$4x - y = 1.$

10. $2x^2 - 3y^2 = 5$
$2x - 3y = 1.$

11. $xy + x - y = 13$
$4x - 3y = 7.$

12. $x^2 - xy + y = 5$
$3x - 2y = 5.$

13. $x^2 + y^2 - 4x + 6y + 3 = 0$
$x + y - 3 = 0.$

14. $x^2 - xy - 3x + 2y = 2$
$3x - 2y - 10 = 0.$

15. $x^2 + 2xy - 2y^2 = 3$
$x - 2y = 3.$

16. $3x^2 + 4xy - y^2 - 2x = 0$
$3x + 4y = 1.$

17. $y^2 - 4xy - 4y + 8x + 3 = 0$
$4x - 2y = 3.$

18. $x^2 + xy - 2y^2 - 3y = 1$
$x - 2y = 1.$

91. A System Without First Degree Terms. If both equations are of the form

$$Ax^2 + Bxy + Cy^2 + F = 0,$$

where the constant term F is not zero, they may be combined to yield two (sometimes only one) linear equations. Each linear equation may then be solved simultaneously with either given equation. The following procedure gives the linear equations:

1. Multiply the members of each equation by a properly chosen constant so that the constant terms of the resulting equations are numerically equal.

2. Then by adding or subtracting eliminate the constant term.

3. Transpose terms, if necessary, to make one member of the equation of step 2 equal to zero. Factor the other member.

4. Set each factor equal to zero. This gives the desired linear equations.

We shall consider systems of equations for which the factors of step 3 are real and rational. However, the linear equations in all cases can be found by solving the equation of this step for one unknown in terms of the other.

If $F = 0$ in one of the given equations, the left member of that equation may be factored to yield the linear equations. The following examples illustrate the process.

Example 1. Solve the system

$$5x^2 - 2xy - y^2 = 2$$
$$2x^2 + xy - 2y^2 = 1.$$

Solution. Leave the first equation unchanged and multiply the members of the second equation by 2. We get

$$\begin{array}{r} 5x^2 - 2xy - y^2 = 2 \\ 4x^2 + 2xy - 4y^2 = 2 \\ \hline x^2 - 4xy + 3y^2 = 0, \end{array}$$

or

$$(x - y)(x - 3y) = 0,$$

and hence

$$x - y = 0 \quad \text{and} \quad x - 3y = 0.$$

We take each of these equations and either of the given equations (preferably the simpler) and form two systems. Thus, choosing the second equation, we have the systems:

$$\begin{array}{rcr} 2x^2 + xy - 2y^2 = 1 & & 2x^2 + xy - 2y^2 = 1 \\ & \text{and} & \\ x - y = 0. & & x - 3y = 0. \end{array}$$

These systems may be solved by the method of Section 90. For the first system the solutions are

$$(1, 1) \text{ and } (-1, -1);$$

and for the second

$$\left(\frac{3\sqrt{19}}{19}, \frac{\sqrt{19}}{19}\right) \quad \text{and} \quad \left(-\frac{3\sqrt{19}}{19}, -\frac{\sqrt{19}}{19}\right).$$

Example 2. Solve the system

$$5x^2 + 2xy - y^2 = 3$$
$$2x^2 - xy - y^2 = 0.$$

Solution. In the second of these equations $F = 0$ and the left member has the real rational factors. We have

$$2x^2 - xy - y^2 = 0,$$
$$(2x + y)(x - y) = 0,$$

and hence

$$2x + y = 0 \quad \text{and} \quad x - y = 0.$$

Taking each of these equations with the equation which we have not used thus far, we have the two systems:

$$\begin{aligned} 5x^2 + 2xy - y^2 &= 3 \\ 2x + y &= 0. \end{aligned} \qquad \text{and} \qquad \begin{aligned} 5x^2 + 2xy - y^2 &= 3 \\ x - y &= 0. \end{aligned}$$

These systems have the following solutions:

$$(i, -2i); \quad (-i, 2i); \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right); \quad \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

Each of these value pairs can be checked in the given equations. Substituting $x = i$ and $y = -2i$ in the right members of the original equations, we obtain

$$5(i)^2 + 2(i)(-2i) - (-2i)^2 = -5 + 4 + 4 = 3. \quad \text{Check.}$$

$$2(i)^2 - (i)(-2i) - (-2i)^2 = -2 - 2 + 4 = 0. \quad \text{Check.}$$

As an exercise, the student might check each of the other value pairs.

Exercise 62

Solve each of the following systems:

1. $2x^2 - 3xy + 2y^2 = 4$
$4x^2 - 6xy + 3y^2 = 4.$

2. $x^2 - xy - y^2 = -5$
$2x^2 + 2xy + y^2 = 10.$

3. $x^2 - 2xy - 2y^2 = 1$
$2x^2 + xy + y^2 = 2.$

4. $x^2 + y^2 - 8 = 0$
$xy + 4 = 0.$

5. $3x^2 - 5xy - 4y^2 = 8$
$x^2 - 2xy - y^2 = 2.$

6. $3xy + 2y^2 = 2$
$3x^2 + xy = 1.$

7. $2x^2 - 2xy - 3y^2 = -7$
$x^2 + 3xy + 2y^2 = 4.$

8. $x^2 - xy - y^2 = 3$
$3x^2 + 4xy + 3y^2 = 6.$

9. $3x^2 - 3xy + y^2 = 7$
$x^2 - 4xy + 4y^2 = 0.$

10. $2x^2 + xy + y^2 = 16$
$2x^2 + 7xy + 6y^2 = 0.$

11. $x^2 + 2xy - y^2 = 14$
$5x^2 - 26xy + 5y^2 = 0.$

12. $4x^2 - 5xy - 6y^2 = 0$
$3x^2 - 3xy - 5y^2 = 9.$

13. $2x^2 - 3xy + 2y^2 = 8$
$3x^2 + 5xy - 4y^2 = 6.$

14. $5x^2 + 2xy + y^2 = 16$
$4x^2 + 4xy = -16.$

15. $6x^2 - 13xy + 6y^2 = 0$
$x^2 + 2xy - 5y^2 = -1.$

16. $5x^2 + 3xy - 2y^2 = 34$
$3x^2 + 8xy - 3y^2 = 0.$

92. Other Methods of Solution. The three methods which we have discussed for solving systems of quadratic equations will be helpful to the student in later courses. There are still other systems, however, whose solutions can be obtained quite simply. The following examples illustrate additional procedures.

Example 1. Solve the system

$$x^2 - 3xy + 2x - 3y = 3$$
$$x^2 + 2xy - 3x + 2y = 8.$$

Solution. We multiply the members of the first equation by 2, those of the second by 3, and add the corresponding members of the resulting equations. This eliminates y. Thus we get

$$\begin{array}{r}
2x^2 - 6xy + 4x - 6y = 6 \\
3x^2 + 6xy - 9x + 6y = 24 \\
\hline
5x^2 \qquad\qquad - 5x \qquad\qquad = 30.
\end{array}$$

The roots of this last equation are $x = -2$ and $x = 3$. By substituting $x = -2$ in either of the given equations, the value $y = 1$ is obtained. The substitution $x = 3$ also yields $y = 1$. Hence the solutions of the given system are $(-2, 1)$ and $(3, 1)$.

Example 2. Solve the system

$$x^2 + y^2 - 2x + 2y = 11$$
$$x^2 + y^2 - 6x - 2y = 7.$$

Solution. By subtracting corresponding members of these equations, we obtain $4x + 4y = 4$, or $x + y = 1$. This linear equation with either of the given equations forms a system with the solutions $(-1, 2)$ and $(4, -3)$.

Example 3. Solve the system

$$x^2 + y^2 + xy - x - y - 10 = 0$$
$$2xy + 3x + 3y + 10 = 0.$$

Solution. Each of these equations would be unaltered by the interchange of x and y. This condition is true because in each equation the coefficients of the first degree terms are equal and the coefficients of the second degree terms are also equal. An equation of this kind is said to be **symmetric** in the unknowns. A system of two symmetric quadratic equations may be solved by first using the substitutions

$$x = u + v \quad \text{and} \quad y = u - v.$$

Making these substitutions and simplifying, we reduce the given equations to

$$3u^2 + v^2 - 2u - 10 = 0$$

$$2u^2 - 2v^2 + 6u + 10 = 0.$$

The unknown v can be eliminated from this system by dividing the members of the last equation by 2 and adding to the preceding equation, member by member. Thus we get the equation

$$4u^2 + u - 5 = 0.$$

The roots of this equation are $u = 1$ and $u = -\frac{5}{4}$. By substituting these values in either of the quadratic equations in u and v, we find the four solutions

$$u = 1,\ v = 3; \quad u = 1,\ v = -3;$$

$$u = -\tfrac{5}{4},\ v = \tfrac{3}{4}\sqrt{5}; \quad u = -\tfrac{5}{4},\ v = -\tfrac{3}{4}\sqrt{5}.$$

Taking each of these solutions and using $x = u + v$ and $y = u - v$, we find the solutions of the original system to be

$$(4, -2),\ (-2, 4),\ (-\tfrac{5}{4} + \tfrac{3}{4}\sqrt{5},\ -\tfrac{5}{4} - \tfrac{3}{4}\sqrt{5}),$$
$$(-\tfrac{5}{4} - \tfrac{3}{4}\sqrt{5},\ -\tfrac{5}{4} + \tfrac{3}{4}\sqrt{5}).$$

Exercise 63

Solve the following systems:

1. $y^2 - 4x + 5y + 14 = 0$
$y^2 + 2x + 2y - 7 = 0.$

2. $4xy + y^2 - 2y - 11 = 0$
$2xy + y^2 - 3y - 4 = 0.$

3. $x^2 + y^2 - 2x - 7 = 0$
$2x^2 - 3y^2 - 6 = 0.$

4. $x^2 - 3xy + x - 3y + 6 = 0$
$x^2 + 2xy + 3x + 2y - 8 = 0.$

5. $3x^2 - y^2 + y - 6 = 0$
$4x^2 - 2y^2 - 3y - 14 = 0.$

6. $x^2 + 2xy - 2y^2 - 3x + 2 = 0$
$x^2 - 4xy + 4y^2 + 2x - 8 = 0.$

7. $x^2 + y^2 + 4x - 6y - 13 = 0$
$x^2 + y^2 - 8x + 2y - 9 = 0.$

8. $x^2 + y^2 - 6x - 1 = 0$
$x^2 + y^2 + 2x + 4y = 5.$

9. $x^2 - 2xy + x - y + 4 = 0$
$2x^2 - 4xy + 2x - 3y + 10 = 0.$

10. $3xy + 4x - 5y = 9$
$2xy + 3x - 4y = 8.$

11. $2x^2 - 5x + 3y + 2 = 0$
$3x^2 - 8x + 3y + 4 = 0.$

12. $x^2 - xy + y^2 - x - 2y = 6$
$x^2 - xy + y^2 - 2x - 3y = 5.$

13. $x^2 + y^2 - 10 = 0$
$xy + x + y - 7 = 0.$

14. $x^2 + y^2 - x - y - 4 = 0$
$2xy - 3x - 3y + 4 = 0.$

15. $x^2 + y^2 + xy + 3x + 3y = 18$
$x^2 + y^2 + 2x + 2y = 15.$

16. $x^2 + y^2 - 10x - 10y + 40 = 0$
$xy - 5x - 5y + 22 = 0.$

Exercise 64

1. The sum of two numbers is 16; the sum of their squares is 130. Find the numbers.

2. The perimeter of a rectangle is 40, and the area is 96. Find the dimensions of the rectangle.

3. The length of a rectangle is 14 units more than its width, and the diagonal is 34 units. Find the lengths of the sides.

4. The perimeter of an isosceles triangle is 36 and altitude is 12. Find the lengths of the sides.

5. The diagonal of a closed rectangular box is 6 inches and the total surface area is 64 square inches. Find the dimensions of the box if the base is a square.

6. The printed area of a rectangular poster is 704 square inches. The printed area plus the area of the margins is 1200 square inches. Find the dimensions of the poster if each of the four margins is 4 inches wide.

7. A rectangular piece of tin has an area of 144 square inches on one side. A two-inch square is cut from each corner and then a box is made by turning up the sides. Find the dimensions of the piece of tin if the box contains 120 cubic inches.

8. A group of children engaged a skating rink for $24. After engaging the rink, 10 more children joined the party, and thereby reduced the cost per child by 20 cents. Find the cost to each child and the final number of children.

9. A man loans a sum of money at simple interest. The interest for one year is $63. Had the interest rate been 1% more, the interest would be $84. Find the amount loaned and the interest rate.

10. A man loaned a sum of money at simple interest. At the end of one year the amount (principal plus interest) was $2884. Had the interest rate been 1% more, the amount would have been $2912. Find the interest rate and the principal.

11. Two pipes can fill a pool in 8 hours. The larger of the pipes can fill the pool in 12 hours less time than the smaller. Find the time required for each pipe alone to fill the pool.

12. A car leaves town A and travels with uniform speed to town B. A second car, traveling 20 miles per hour faster than the first car, goes from town B to town A. The two cars start at the same time and meet in 1 hour, and one of the cars makes the drive, from town to town, in 50 minutes less time than is required for the other car. Find the speed of each car and the distance between the towns.

13. Three airplanes take off at the same time from an airfield and travel in straight courses to another field. The first airplane flies 40 miles per hour faster than the second airplane and makes the flight in 40 minutes less time. The third airplane flies 40 miles per hour slower than the second airplane and consumes 1 hour more flight time than the second one. Find the speed of the second airplane and the distance between the fields.

Chapter Twelve

PROGRESSIONS

93. Introduction. A succession of numbers which are determined by some given law is called a **sequence.** Each number is called a **term.** The terms are ordered as the first, second, third, and so on. The numbers

$$1, 4, 9, 16, 25, 36$$

constitute a sequence of six terms. As may be seen, the terms are formed by squaring the integers 1 through 6.

A great variety of sequences is used in mathematics. However, we shall study only two kinds which, though comparatively simple, are of especial importance.

94. Arithmetic Progressions. A sequence in which each term after the first is formed by adding a fixed number to the preceding term is called an **arithmetic progression.** The number which is added in this way is called the **common difference.**

Illustrations:

$5, 7, 9, \ 11, \ 13$ common difference 2

$7, 4, 1, -2, -5$ common difference -3

If we let a stand for the first term and d for the common difference of an arithmetic progression, the successive terms may be written by repeated additions of d. Thus the first four terms are

$$a, \quad a + d, \quad a + 2d, \quad a + 3d.$$

The coefficients of d increase by one as we go from term to term. By noticing also that each coefficient is one less than the number of the term, we can find any term without writing the preceding terms. Thus we have

$$\begin{aligned} \text{8th term} &= a + 7d, \\ \text{20th term} &= a + 19d, \\ n\text{th term} &= a + (n - 1)d. \end{aligned}$$

Designating the nth term by l, we have the formula

$$l = a + (n - 1)d.$$

This formula is frequently used to find the last term of an arithmetic progression. But, with a and d given, it will yield any desired term by substituting the proper positive integer for n.

Example 1. Find the 30th term of 2, 6, 10, $\cdots$.

Solution. Here $a = 2$, $d = 4$, and $n = 30$. Hence, using the formula, we find

$$l = 2 + (30 - 1)4 = 2 + 116 = 118.$$

Example 2. Find the 21st term of $\frac{9}{4}$, 2, $\frac{7}{4}$, $\frac{3}{2}$, $\cdots$.

Solution. The common difference, found by subtracting a term from the following term, is negative in this progression. We use the values $a = \frac{9}{4}$, $d = -\frac{1}{4}$, $n = 21$, and get

$$l = \tfrac{9}{4} + 20(-\tfrac{1}{4}) = \tfrac{9}{4} - \tfrac{20}{4} = -\tfrac{11}{4}.$$

The first and last terms of an arithmetic progression are called **extremes.** The terms between the first and last are called **arithmetic means.** If we wish to insert a certain number of arithmetic means between two given numbers, we may employ the formula for the nth term to find d. Then the progression may be written by repeated additions of d.

Example 3. Insert 5 arithmetic means between 7 and 25.

Solution. $a = 7$, $l = 25$. In order to get the number of terms, we notice that there are 5 means and 2 extremes, giving $n = 7$. Using the formula $l = a + (n - 1)d$, we have

$$25 = 7 + 6d$$

and

$$d = 3.$$

The progression is 7, 10, 13, 16, 19, 22, 25.

Exercise 65

Determine which of the following sequences are arithmetic progressions. Write the next two terms of each arithmetic progression.

1. 4, 7, 10, 13.
2. 8, 6, 4, 2.
3. $6\frac{1}{2}$, 8, $9\frac{1}{2}$, 11.
4. 0, 4, 8, 12.
5. 1.1, 2.3, 3.4.
6. -9, -5, -1, 3.
7. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$.
8. 0, 3, 7, 11.
9. 2, 4, 8, 16.
10. $4\frac{2}{3}$, $3\frac{1}{3}$, 2, $\frac{2}{3}$.
11. 3.8, 3.1, 2.4.
12. 2.3, 1.4, 0.5.
13. $2x - 5y$, $4x - 7y$, $6x - 9y$.
14. $5x - 14y$, $3x - 4y$, $x + 6y$

Find the term indicated:

15. 5, 7, 9, ⋯ 25th. **16.** 6, 4, 2, ⋯ 18th. **17.** 52, 47, 42, ⋯ 11th.

18. -6, -2, 2, ⋯ 16th. **19.** 27, 24, 21, ⋯ 51st.
20. 1.3, 2.5, 3.7, ⋯ 21st. **21.** $\frac{33}{4}$, $\frac{29}{4}$, $\frac{25}{4}$, ⋯ 15th.
22. $7\frac{1}{3}$, $8\frac{1}{12}$, $8\frac{5}{6}$, ⋯ 22nd. **23.** 8.9, 7.8, 6.7, ⋯ 52nd.

24. x, $2x + 1$, $3x + 2$, ⋯ 10th.
25. $2x - y$, $3x - 2y$, $4x - 3y$, ⋯ 12th.

Insert the indicated number of means between each pair of numbers:

26. 5 and 23; 7 means. **27.** 2 and 25; 5 means. **28.** 6 and 15; 1 mean.

29. 8 and 4; 3 means. **30.** -15 and 15; 5 means.

31. 6 and 20; 3 means.

95. The Sum of *n* Terms of an Arithmetic Progression. We wish now to derive a formula for the sum of the terms of an arithmetic progression. If we start with the first term a and add d repeatedly, a progression of any desired number of terms may be written. By starting with the last term l and subtracting d repeatedly, the same terms are obtained in reverse order. Using five terms, for example, and denoting their sum by S, we have the equations

$$S = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d),$$
$$S = l + (l - d) + (l - 2d) + (l - 3d) + (l - 4d).$$

Adding the corresponding sides of the two equations, we get

$$\begin{aligned} 2S &= (a + l) + (a + l) + (a + l) + (a + l) + (a + l) \\ &= 5(a + l). \end{aligned}$$

We see that the sum of the first and last terms multiplied by the number of terms gives twice the sum of the five terms. Now it is evident that $6(a + l)$ is twice the sum of 6 terms, $7(a + l)$ is twice the sum of 7 terms, and, in general, $n(a + l)$ is twice the sum of n terms. Hence we have for the sum S of n terms

$$S = \frac{n}{2}(a + l).$$

We now have two useful formulas which are applicable to arithmetic progressions. The formulas, rewritten for convenient reference, are

$$\boldsymbol{l = a + (n - 1)d} \tag{1}$$

$$\boldsymbol{S = \frac{n}{2}(a + l).} \tag{2}$$

The five quantities a, l, d, n, and S, occurring in the formulas, are called the **elements** of an arithmetic progression. If three of the elements are given, the other two may be found by the use of formulas (1) and (2). When the two unknowns appear in both formulas, it is necessary to use them as a system of two equations in two unknowns. In case the three given elements appear in one of the formulas, that formula will yield a fourth element, and then the other formula gives the remaining element.

Example 1. Find the sum of 20 terms of the arithmetic progression $3, 5, 7, \cdots$.

Solution. The known elements are $a = 3$, $d = 2$, and $n = 20$. We can not apply formula (2) directly because it contains two unknowns. Formula (1), however, has only one unknown and we use it to find l. Thus

$$l = 3 + 19(2) = 3 + 38 = 41.$$

Having found l, we use formula (2) to yield

$$S = \tfrac{20}{2}(3 + 41) = 10(44) = 440.$$

Example 2. Given $n = 13$, $l = 51$, and $d = 4$, find a and S.

Solution. Formula (1) contains only one of the unknowns. Substituting in it, we obtain

$$51 = a + 12(4),$$

$$a = 3.$$

The remaining unknown S can now be found by using formula (2). Thus we get

$$S = \tfrac{13}{2}(3 + 51) = 351.$$

Example 3. Given $n = 11$, $S = 77$, and $d = 3$, find a and l.

Solution. The unknowns a and l occur in both formulas. Substituting in both, we obtain the linear system

$$l = a + 10(3)$$

$$77 = \tfrac{11}{2}(a + l),$$

which may be written as

$$a - l = -30$$

$$a + l = 14.$$

The solution is $a = -8$, $l = 22$. The progression may now be written as $-8, -5, -2, 1, 4, \cdots, 22$.

Example 4. Given the elements $d = 3$, $l = 33$, and $S = 195$, find a and n.

Solution. Substituting the known elements in the formulas gives the system

$$33 = a + (n - 1)3$$

$$195 = \frac{n}{2}(a + 33),$$

which reduces to

$$a + 3n = 36$$

$$n(a + 33) = 390.$$

We solve the first equation for a and substitute the result in the second equation. Thus we have

$$n(36 - 3n + 33) = 390$$

$$69n - 3n^2 = 390$$

$$n^2 - 23n + 130 = 0$$

$$(n - 10)(n - 13) = 0,$$

$$n = 10 \text{ and } 13.$$

The corresponding values of a, found by substituting for n in either equation of the system, are 6 and -3. Hence the two solutions are $a = 6$, $n = 10$ and $a = -3$, $n = 13$. The two progressions having the three given elements are

$$6, 9, 12, \cdots, 33$$

and

$$-3, 0, 3, 6, 9, 12, \cdots, 33.$$

When a and n are the unknowns, as in the preceding example, and when l and n are the unknowns, formulas (1) and (2) give a linear equation and a second degree equation. This means that there may be two solutions. A value for n other than a positive integer, however, must be rejected.

Exercise 66

Find l and S for each arithmetic progression:

1. $2, 6, 10, \cdots$ to 18 terms.

2. $1, 3, 5, \cdots$ to 23 terms.

3. $-37, -34, -31, \cdots$ to 21 terms.

4. $18, 13, 8, \cdots$ to 15 terms.

5. $-12, -8, -4, \cdots$ to 7 terms.

6. $5.7, 3.6, 1.5, \cdots$ to 14 terms.

7. $\frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \cdots$ to 10 terms.

8. $\frac{13}{3}, 6, \frac{23}{3}, \cdots$ to 17 terms.

Find the two missing elements in each arithmetic progression:

9. $a = -22, l = 14, n = 10.$
10. $a = \frac{3}{2}, l = 34, d = \frac{5}{2}.$
11. $l = 40, d = 4, n = 10.$
12. $a = 5, n = 10, S = 90.$
13. $l = 50, n = 16, S = 200.$
14. $a = 3, l = 21, S = 156.$
15. $d = 4, n = 37, S = 148.$
16. $l = -15, d = -2, S = -48.$
17. $a = 12, d = -3, S = -54.$
18. $d = -3, n = 10, S = -35.$
19. $l = 8, d = 3, S = -20.$
20. $d = 4, l = 28, S = 108.$

21. Find the sum of the integers from 1 to 50, inclusive.

22. Find the sum of the even integers between 19 and 71.

23. Find the sum of the integers between 20 and 80 which are divisible by 3.

24. Find the sum of the integers between 6 and 141 which are divisible by 7.

25. There are 28 logs in the bottom layer of a pile of logs, 27 in the next layer, 26 in the next, and so on to the top layer which has 1 log. Find the number of logs in the pile.

26. In a lottery, tickets are numbered consecutively from 1 to 80. The tickets are drawn by 80 persons, each of whom pays as many cents as the number on his ticket. Find the total amount which is paid for the tickets. What is the least number of tickets which, under this plan, would bring in as much as $100?

27. A body falling from rest travels 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on. Find how far the body falls (a) during the 8th second; (b) during the first 8 seconds.

28. An object, dropped from a height of 3600 feet above the earth, falls 16 feet the first second, 48 feet the second second, 80 feet the third second, and so on. Find the time required for the object to reach the earth.

29. An employer engages a man at a beginning salary of $3600 per year. If the salary is increased $200 yearly, what would be the total earnings over a period of 12 years?

30. A man borrows $10,000. He agrees to pay $1000 at the end of each year for 10 years, and also to pay at the end of each year 4% interest on the amount he owed during the year. Find how much interest will be paid in settling the debt.

31. A man buys a car priced at $3000. He agrees to pay each month, beginning one month after the purchase date, $100 on the debt plus an interest charge of 1% of the debt outstanding during the month. Find the total amount of interest which he will pay.

32. An employer engages a man at a beginning salary of $4000 per year. He offers to increase the salary $200 per year at the end of each year or, alternatively, he will increase the pay $50 for each six months at the end of each half year. Which is the more attractive proposition?

33. Prove that the sum of the first n positive even integers is $n(n + 1)$.

34. Prove that the sum of the first n positive odd integers is n^2.

35. Solve the system $l = a + (n - 1)d, S = \frac{n}{2}(a + l)$ for a and l.

36. Solve $l = a + (n - 1)d$ for each element in terms of the other three.

96. Geometric Progressions. A sequence in which each term after the first is a fixed number times the preceding term is a **geometric progression.** The fixed number is called the **common ratio.**

Illustrations:

$2,\ 4,\ 8,\ 16$	common ratio 2
$1,\ -3,\ 9,\ -27$	common ratio -3
$6,\ 4,\ \frac{8}{3},\ \frac{16}{9}$	common ratio $\frac{2}{3}$

The first term, the common ratio, the number of terms, the nth or last term, and the sum of n terms constitute the five **elements** of a geometric progression. We denote the elements, in the order named, by a, r, n, l, and S.

The first and last terms of a geometric progression are called the **extremes** and the terms between are called the **means.**

97. The nth Term and the Sum of n Terms of a Geometric Progression. If a is the first term and r is the common ratio, we can form a geometric progression of any chosen number of terms. Thus, for five terms, we write

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4.$$

By noticing that the exponent of r in each term is one less than the number of the term, we see that the nth term, designated by l, is

$$l = ar^{n-1}.$$

Example 1. Find the 6th term of the progression $9, -6, 4, \cdots$.

Solution. In this case we see that the first term multiplied by $-\frac{2}{3}$ gives the second term, and the product of the second term and $-\frac{2}{3}$ is the third term. Using $r = -\frac{2}{3}$ and $a = 9$, we have

$$l = 9(-\tfrac{2}{3})^5 = 9(-\tfrac{32}{243}) = -\tfrac{32}{27}.$$

To derive a formula for the sum S of n terms of a geometric progression, we write the two equations

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} \\ rS &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \end{aligned}$$

The second equation is obtained by multiplying each member of the first by r and placing the like terms of the right members one under the other.

By subtracting the members of the lower equation from corresponding members of the upper equation, we get

$$S - rS = a - ar^n,$$

$$S(1 - r) = a(1 - r^n),$$

$$S = \frac{a(1 - r^n)}{1 - r}.$$

This formula for the sum is in terms of a, r, and n. The sum in terms of a, r, and l can be obtained by using the formula $l = ar^{n-1}$. Multiplying both members by r gives $rl = ar^n$, and we may write

$$S = \frac{a - ar^n}{1 - r} = \frac{a - rl}{1 - r}.$$

We now have the following formulas involving the elements of geometric progressions:

$$\boldsymbol{l = ar^{n-1}} \tag{1}$$

$$\boldsymbol{S = \frac{a(1 - r^n)}{1 - r}} \tag{2}$$

$$\boldsymbol{S = \frac{a - rl}{1 - r}.} \tag{3}$$

With any three elements known, these formulas suffice to yield the remaining elements. In some cases there will be more than one solution.

Example 2. Find the sum of the progression $\frac{1}{2}$, 1, 2, $\cdots$ to 8 terms.

Solution. The known elements are $a = \frac{1}{2}$, $r = 2$, and $n = 8$. Substituting these values in formula (2) gives

$$S = \frac{\frac{1}{2}(1 - 2^8)}{1 - 2} = \frac{\frac{1}{2}(1 - 256)}{-1} = 127\tfrac{1}{2}.$$

Example 3. Find 8th term and the sum of 8 terms of the progression 81, -27, 9, $\cdots$.

Solution. Here $a = 81$, $r = -\frac{1}{3}$, and $n = 8$. Substituting first in formula (1), we get

$$l = 81\left(-\frac{1}{3}\right)^7 = -\frac{3^4}{3^7} = -\frac{1}{3^3} = -\frac{1}{27}.$$

With a, r, and l now known, we substitute in formula (3) and obtain

$$S = \frac{81 - (-\frac{1}{3})(-\frac{1}{27})}{1 - (-\frac{1}{3})} = \frac{81 - \frac{1}{81}}{\frac{4}{3}} = \frac{6560}{81} \cdot \frac{3}{4} = \frac{1640}{27} = 60\tfrac{20}{27}.$$

Example 4. Given $l = 80$, $n = 3$, and $S = 65$, find a and r.

Solution. Both unknowns occur in each formula. Therefore, we use formulas (1) and (2) together and solve for n and r. From (1),

$$80 = ar^2 \quad \text{or} \quad a = \frac{80}{r^2}.$$

From (2),

$$65 = \frac{a(1 - r^3)}{1 - r} = a(1 + r + r^2).$$

Substituting for a in this equation gives

$$65 = \frac{80}{r^2}(1 + r + r^2),$$

$$65r^2 = 80 + 80r + 80r^2,$$

$$15r^2 + 80r + 80 = 0,$$

$$3r^2 + 16r + 16 = 0,$$

$$(3r + 4)(r + 4) = 0,$$

$$r = -\tfrac{4}{3} \text{ and } -4.$$

Then

$$a = \frac{80}{r^2} = 45 \text{ and } 5.$$

The solutions are $a = 45$, $r = -\frac{4}{3}$ and $a = 5$, $r = -4$. The two progressions with the three given elements are

$$45, -60, 80 \text{ and } 5, -20, 80.$$

Exercise 67

Find l and S for each geometric progression:

1. $3, 6, 12, \cdots$ to 7 terms.
2. $18, 9, 4\frac{1}{2}, \cdots$ to 8 terms.
3. $20, -10, 5, \cdots$ to 7 terms.
4. $18, 12, 8, \cdots$ to 7 terms.
5. $48, 36, 27, \cdots$ to 8 terms.
6. $16, -12, 9, \cdots$ to 6 terms.

Find S for each progression determined by the given elements:

7. $a = 3, r = \frac{1}{2}, l = \frac{3}{64}$.
8. $a = 2, r = -2, l = -64$.
9. $a = 6, r = 3, l = 486$.
10. $a = 27, r = -\frac{1}{3}, l = \frac{1}{27}$.
11. $a = -6, r = -\frac{3}{2}, l = -\frac{243}{8}$.
12. $a = 32, r = \frac{3}{2}, l = 243$.

Find r and insert the indicated number of geometric means between each pair of numbers:

13. 3 and 48; 3 means.
14. 9 and $\frac{1}{9}$; 3 means.
15. $\frac{1}{4}$ and -8; 4 means.
16. $\frac{27}{4}$ and $\frac{8}{9}$; 4 means.
17. $\frac{25}{9}$ and $\frac{27}{125}$; 4 means.
18. -1 and 128; 6 means.

Find the two missing elements in each geometric progression:

19. $l = 81$, $r = 3$, $n = 6$.
20. $a = \frac{2}{9}$, $r = 3$, $l = 18$.
21. $a = \frac{1}{8}$, $l = \frac{27}{64}$, $n = 4$.
22. $a = 15$, $r = \frac{2}{5}$, $S = \frac{117}{5}$.
23. $S = 21$, $a = 16$, $n = 3$.
24. $S = -26$, $r = -\frac{3}{2}$, $n = 4$.
25. $S = \frac{62}{3}$, $n = 3$, $l = \frac{2}{3}$.
26. $S = 13$, $r = -\frac{2}{3}$, $l = -8$.
27. $a = 1$, $l = \frac{27}{8}$, $S = \frac{65}{8}$.
28. $S = 18$, $l = 24$, $n = 3$.

29. If at the end of each year the value of a car is $\frac{2}{3}$ of its value at the beginning of the year, find the value of $3000 car after 4 years.

30. The number of bacteria in a certain culture doubles in 3 hours. If there are N bacteria to start with, find the number in 24 hours.

31. If the population of a city is 2,000,000 and if the rate of increase is 10% each year, what will the population be at the end of 6 years?

32. With each stroke a vacuum pump removes 10% of the remaining air from a container. How much of the original air is in the container after 5 strokes? How many strokes are required to remove 90% of the air?

33. If the distance of each succeeding swing of a pendulum is 0.9 of the previous swing and the distance of the first swing is 5 inches, find the distance of the 7th swing. Find the total distance of the 7 swings.

34. A man deposits $200 in a savings bank. Interest at 3% annually is added to the principal at the end of each year. How much is in the account at the end of 5 years?

35. A certain ball when dropped from a height rebounds $\frac{2}{3}$ of the distance from which it fell. Find the total distance traveled by the ball from the time it is dropped at a height of 30 feet until it strikes the ground the 5th time.

36. A ball is thrown vertically upward to a height of 100 feet. Each time the ball strikes the ground it rebounds to a height of $\frac{1}{3}$ the previous height. How far has the ball traveled up and down, when it strikes the ground the 6th time?

37. According to a story, a farmer and blacksmith made the following agreement: for shoeing his horse the farmer was to pay the blacksmith 1 cent for the first nail, 2 cents for the second nail, 4 cents for the third nail, 8 cents for the fourth nail, and so on. How much did the farmer owe if 32 nails were used?

38. A 10-quart container is filled with water. One quart of water is drained out and replaced with a quart of alcohol. After mixing, a quart of the solution is drained out and replaced with alcohol. This process is continued until 5 quarts of alcohol have been put in the container. The solution in the container is then what percent alcohol?

39. A man has $10,000 in a bank. Each month he withdraws $\frac{1}{3}$ of the amount remaining to his credit. How much does he have left after 6 withdrawals?

40. A man can have a position at a beginning salary of \$4000 per year with an increase of \$450 per year each year, or he can begin at the same annual pay and each following year receives 10% more than during the preceding year. Which proposition will yield the better income during the 5th year? Which proposition will yield the better income for the period of 5 years?

98. Infinite Geometric Progressions. In dividing 1 by 3 we obtain the unending decimal fraction $0.3333\cdots$. By stopping at one, two, or three decimal places approximations are obtained. The approximation is better as more decimal places are kept. The digits in the fraction are terms of a geometric progression in which $r = 0.1$. This is evident when we write

$$\tfrac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots.$$

The numbers on the right are terms of an unending, or infinite, geometric progression. Although the progression does not end, we shall say that its sum is $\frac{1}{3}$. The reason for taking this value as the sum is that in adding enough terms we can obtain a result as close to $\frac{1}{3}$ as we please.

Next let us consider any geometric progression in which the common ratio is numerically less than 1. In this case the terms decrease numerically, and, by going far enough into the progression, terms can be found as near zero as we please. This fact may be seen intuitively by examining the formula for the nth term, $l = ar^{n-1}$. If r is between -1 and 1, its square, cube, fourth power, and so on, diminish in absolute value and approach zero.

Illustrations:

$$(\tfrac{2}{3})^2 = \tfrac{4}{9} \qquad (-0.1)^2 = 0.01$$

$$(\tfrac{2}{3})^3 = \tfrac{8}{27} \qquad (-0.1)^3 = -0.001$$

$$(\tfrac{2}{3})^4 = \tfrac{16}{81} \qquad (-0.1)^4 = 0.0001$$

$$(\tfrac{2}{3})^5 = \tfrac{32}{243} \qquad (-0.1)^5 = -0.00001$$

In the formula for the sum of n terms

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r},$$

we notice that ar^n can be made arbitrarily near zero if r is between -1 and 1 and if n is sufficiently large. This means that the last fraction on the right can also be brought as near zero as we please. Hence the sum of terms, for a large n, is approximately equal to the first fraction in the right member of the equation. Moreover, the approximation improves as more and more terms are used. We say then that

$$S = \frac{a}{1 - r}$$

is the **sum** of an infinite geometric progression if r is between -1 and 1.

Example 1. Find the sum of the infinite geometric progression 1, $\frac{2}{3}$, $\frac{4}{9}$, $\cdots$.

Solution. In this progression $a = 1$ and $r = \frac{2}{3}$. Hence

$$S = \frac{1}{1 - \frac{2}{3}} = 3.$$

Example 2. Express the repeating decimal 4.7676 $\cdots$ as a rational number.

Solution. The fraction 0.7676 may be written in the form

$$0.76 + 0.0076 + 0.000076 + 0.00000076 + \cdots.$$

We see that $a = 0.76$, $r = 0.01$, and therefore

$$S = \frac{0.76}{1 - 0.01} = \frac{0.76}{0.99} = \frac{76}{99}.$$

Hence

$$4.7676 = 4\tfrac{76}{99} = \tfrac{472}{99}.$$

Exercise 68

Find the sum of each infinite geometric progression:

1. 1, $\frac{1}{3}$, $\frac{1}{9}$, $\cdots$.
2. $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{1}{8}$, $\cdots$.
3. 8, 2, $\frac{1}{2}$, $\cdots$.
4. 8, -1, $\frac{1}{8}$, $\cdots$.
5. 10, 2, $\frac{2}{5}$, $\cdots$.
6. -36, 6, -1, $\cdots$.
7. 0.5, 0.05, 0.005, $\cdots$.
8. 0.1, 0.05, 0.025, $\cdots$.
9. 7, 2.1, 0.63, $\cdots$.
10. 8, -1.6, 0.32, $\cdots$.
11. -36, 6, -1, $\cdots$.
12. 1, -0.7, 0.49, $\cdots$.

Express the following repeating decimals as rational fractions in lowest terms:

13. 0.888$\cdots$.
14. 0.555$\cdots$.
15. 0.777$\cdots$.
16. 0.3434$\cdots$.
17. 0.2121$\cdots$.
18. 0.4747$\cdots$.
19. 3.1212$\cdots$.
20. 2.8383$\cdots$.
21. 1.1919$\cdots$.
22. 0.123123$\cdots$.
23. 0.567567$\cdots$.
24. 0.819819$\cdots$.
25. 0.142857142857$\cdots$.
26. 0.076923076923$\cdots$.
27. 3.17121712$\cdots$.

28. Compare the sum of the infinite geometrical progression 1, $\frac{1}{2}$, $\frac{1}{4}$, $\cdots$ with the sum of the first 10 terms.

29. Compare the sum of the infinite geometrical progression 1, $-\frac{1}{9}$, $\frac{1}{81}$, $\cdots$ with the sum of the first 4 terms.

30. Compare the sum of the infinite progression 1, 0.05, 0.0025, $\cdots$ with the sum of the first two terms.

31. A certain ball rebounds $\frac{4}{5}$ of the height from which it has fallen. Find how far the ball travels before coming to rest if it is thrown vertically upward to a height of 60 feet.

32. Find the total distance which the tip of a pendulum will travel if the first swing is 10 inches, and each succeeding swing is $\frac{9}{10}$ of the previous one.

Chapter Thirteen

INEQUALITIES

99. Definition of Inequalities. A statement that one algebraic expression is greater than, or less than, another algebraic expression is called an **inequality.** The expressions, called the **members,** must be real numbers. The usual symbols of inequalities are $>$ and $<$, and are read, respectively, "is greater than" and "is less than." The statement that a is greater than b may be expressed by $a > b$. Similarly, $c < d$ means c is less than d. The symbol opens to the greater member and points to the smaller.

Two inequalities with their inequality signs pointing in the same direction are of the **same sense.** If the signs point in opposite directions, the inequalities have **unlike senses.**

Illustrations: $x + 1 > 5$ and $x^2 - 3 > 4x$ have the same sense, but $3x - 2 < x^2$ and $x^3 - 3 > 0$ have unlike senses.

100. Properties of Inequalities. Just as there are operations which may be performed upon the members of an equation, there are similar operations by which inequalities may be transformed. We shall consider three operations or properties.

Property I. *The sense of an inequality is not changed if the same number is added to, or subtracted from, both members; or, stated in symbols,*

$$\textit{if} \quad a > b, \quad \textit{then} \quad a + c > b + c.$$

Proof. Since a is greater than b, it is necessary to add some positive number n to b in order to have

$$a = b + n.$$

Add c:

$$a + c = b + c + n.$$

Now, since a positive number n added to $b + c$ yields $a + c$, it follows that

$$a + c > b + c.$$

Hence c, a positive or a negative number, may be added to both members of the inequality without changing the sense of the inequality.

From Property I it follows immediately that terms of an inequality may be transposed just as in an equation.

Property II. *The sense of an inequality is not changed if both members are multiplied by, or divided by, the same positive number. That is,*

$$\text{if} \quad a > b, \quad \text{then} \quad ac > bc \quad \text{and} \quad \frac{a}{c} > \frac{b}{c},$$

where c is a positive number.

Proof: Since $a > b$, b plus some positive number n will yield a, or

$$a = b + n.$$

Multiply by c:

$$ac = bc + nc.$$

The term nc is positive since n and c are positive. Hence

$$ac > bc.$$

Next, we divide $a = b + n$ by c, and get $\frac{a}{c} = \frac{b}{c} + \frac{n}{c}\cdot$ Since $\frac{n}{c}$ is positive, it follows that $\frac{a}{c} > \frac{b}{c}\cdot$

Property III. *The sense of an inequality is reversed if both members are multiplied by, or divided by, the same negative number. In symbols,*

$$\text{if} \quad a > b, \quad \text{then} \quad ac < bc, \quad \text{and} \quad \frac{a}{c} < \frac{b}{c},$$

where c is negative.

Proof: We have

$$a = b + n,$$

where n, is positive.
Multiply by c:

$$ac = bc + nc.$$

The term nc is negative since n is positive and c is negative. Hence

$$ac < bc.$$

Dividing by c^2 gives

$$\frac{ac}{c^2} < \frac{bc}{c^2} \quad \text{or} \quad \frac{a}{c} < \frac{b}{c}\cdot$$

Though c is negative, c^2 is positive; and the division does not reverse the sense of the inequality.

101. Absolute and Conditional Inequalities. An inequality which holds for all defined values of the letters involved is called an **absolute inequality.** An inequality which holds for only certain of the defined values of the letters involved is a **conditional inequality.**

Illustrations: The inequalities

$$a^2 + b^2 > -1,$$

and

$$x^2 + 2x + 3 > 0$$

are absolute inequalities. The first obviously holds for all real values of a and b. The left member of the second may be written as

$$(x + 1)^2 + 2$$

which, of course, is positive since $(x + 1)^2$ is not negative for any real value of x. But the inequality

$$x + 3 < 4$$

is conditional; it holds only if $x < 1$, and not for all values of x.

Example 1. If a and b are unequal real numbers, show that

$$a^2 + b^2 > 2ab.$$

Proof: Since a and b are unequal, $a - b$ is not zero, and

$$(a - b)^2 > 0.$$

Then

$$a^2 - 2ab + b^2 > 0,$$

and, transposing,

$$a^2 + b^2 > 2ab.$$

To **solve** an inequality means to find the values of the unknown (or unknowns) for which the inequality is valid. Those values constitute the solution.

Example 2. Solve the inequality $2 - 3x < 2x + 12$.
Solution. Transpose 2 and $2x$:

$$-5x < 10.$$

Divide by -5 and reverse the inequality sign (Property III):

$$x > -2.$$

The given inequality is conditional; it holds only for x greater than -2.

Example 3. Solve the inequality

$$2x^2 - x - 15 < 0.$$

Solution. The first step is to factor the left side. Thus we have

$$(2x + 5)(x - 3) < 0 \quad \text{or} \quad 2(x + \tfrac{5}{2})(x - 3) < 0.$$

In the factored form we see that the zeros of $2x^2 - x - 15$ are $x = -\frac{5}{2}$ and $x = 3$. For all other values of x the left side is either positive or negative. It is negative when the factors $x + \frac{5}{2}$ and $x - 3$ are of different signs; and positive when the factors have the same signs. The factor $x + \frac{5}{2}$ is negative when $x < -\frac{5}{2}$ and positive when $x > -\frac{5}{2}$. This is pictured on the number scale (Fig. 18) by the minus signs to the left of $-\frac{5}{2}$ and the plus signs to the right. Similarly the factor $x - 3$ is negative to the left of 3 on the scale and positive to the right. From the diagram it is easy to see that the factors have opposite signs for values of x between $-\frac{5}{2}$ and 3. This solution may be expressed in the form

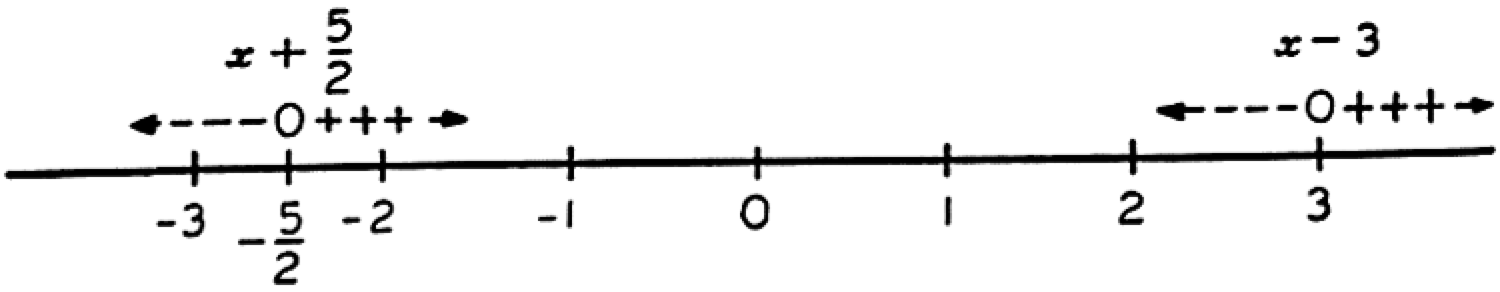


FIG. 18

$$-\tfrac{5}{2} < x < 3.$$

This result is shown graphically in Figure 19, where we have drawn the

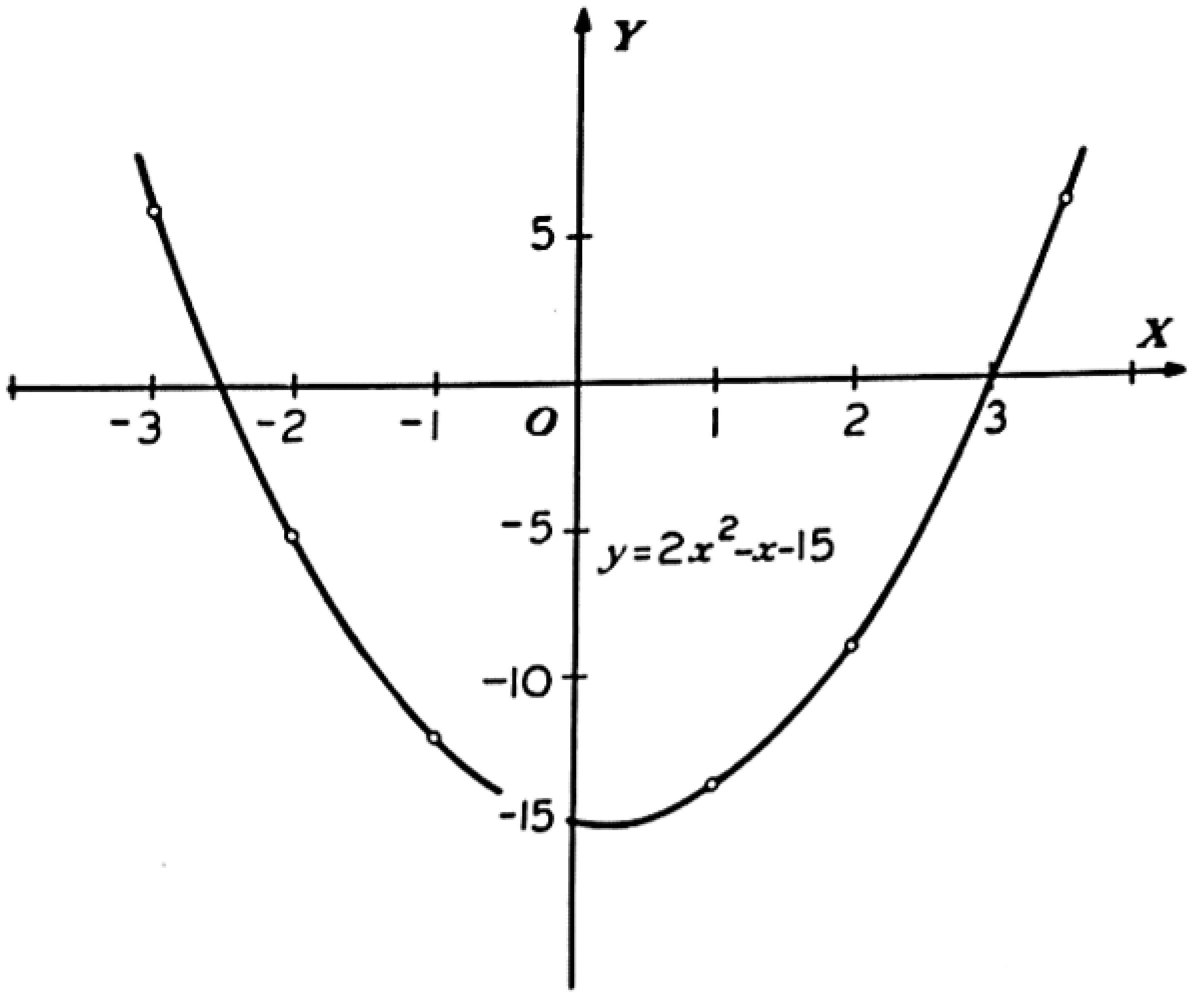


FIG. 19

curve for the function $2x^2 - x - 15$. The graph is below the x-axis for values of x between $-\frac{5}{2}$ and 3.

Example 4. Solve the inequality $(x + 3)(x - 1)(x - 4) > 0$.

Solution. The inequality holds when two of the factors in the left member are negative and the other positive, and also holds when the three factors are positive. The zeros of the left member are $x = -3$, $x = 1$, and $x = 4$. Each factor is placed by its zero point on the number scale (Fig. 20). The minus and plus signs indicate that a factor is negative for values of x to the left of its zero point and positive to the right. Referring to the diagram, we can easily pick out the values of x which satisfy the given inequality. Values of x between -3 and 1 make $x + 3$ positive and both $x - 1$ and $x - 4$ negative. For these values the product of the three factors is positive. All three factors are positive for values of x greater than 4. Hence the solution is given by

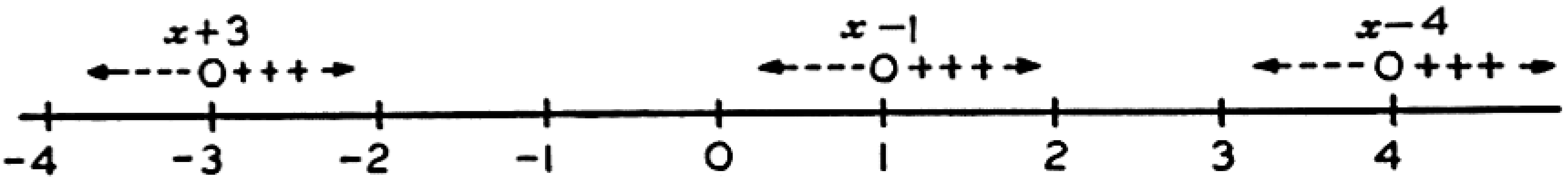


FIG. 20

$$-3 < x < 1 \quad \text{and} \quad x > 4.$$

Example 5. Solve the inequality $|2x - 3| < 5$.

Solution. The vertical bars are used to indicate the absolute value (Sec. 2) of the enclosed expression. Hence the inequality means that $2x - 3$ is between -5 and 5, or the inequality $2x - 3 > -5$ and the inequality $2x - 3 < 5$ are to be satisfied. These inequalities can be written as

$$-5 < 2x - 3 < 5.$$

By adding 3 to each member and then dividing by 2, we obtain

$$-2 < 2x < 8, \quad -1 < x < 4.$$

Thus we see that the given inequality is valid for all values of x between -1 and 4.

Exercise 69

Establish the following inequalities if the letters stand for unequal positive numbers:

1. $a^2 + b^2 < (a + b)^2$.
2. $(x + y)^3 > x^3 + y^3$.
3. $(x - y)^2 < x^2 + y^2$.
4. $\dfrac{a^2 + b^2}{a + b} > a - b$.
5. $\dfrac{x}{y} + \dfrac{y}{x} > 2$.
6. $\dfrac{a + b}{2} > \sqrt{ab}$.

7. $\dfrac{x^3 + y^3}{x + y} > xy.$

8. $\dfrac{2ab}{a + b} < \sqrt{ab}.$

9. $\dfrac{x + y}{4x} > \dfrac{y}{x + y}.$

10. $(x + y + z)^2 > x^2 + y^2 + z^2.$

11. $a^3 + b^3 > a(a^2 - ab + b^2).$

Solve the following inequalities:

12. $3x > 9.$
13. $2x - 3 < 5.$
14. $5x - 1 > 3x + 7.$
15. $6x + 3 < x - 9.$
16. $x + 6 < 4 - 3x.$
17. $4 > 3x + 5.$
18. $8x - 1 > 6x + 4.$
19. $5x - 7 < 3x + 2.$
20. $x + 1 > 3x - 4.$

21. $x^2 > 4.$
22. $x^2 < 25.$
23. $x^2 - 16 > 0.$
24. $x(x - 1) < 0.$
25. $x^2 - 4x + 3 < 0.$
26. $x^2 - x - 6 < 0.$
27. $x^2 + 3 > 0.$
28. $x^2 + 36 < 0.$
29. $x^2 > 0.$
30. $x^2 + 2x + 2 > 0.$
31. $x^2 - 4x + 5 > 0.$
32. $x^2 - 6x + 10 > 0.$
33. $2x^2 - x - 10 < 0.$
34. $3x^2 - 4x - 4 < 0.$
35. $2x^2 - 9x + 7 < 0.$
36. $(x - 1)^2(x - 2) > 0.$
37. $(x - 2)^2(x - 3) < 0.$
38. $(x + 2)^2(x - 5) < 0.$

39. $(x - 1)(x - 3)(x - 4) > 0.$
40. $(x + 1)(x - 2)(x + 3) < 0.$
41. $(x - 3)(x + 5)(x - 6) > 0.$
42. $(x + 4)(x + 5)(x - 4) < 0.$

43. $|x - 4| < 6.$
44. $|x| > 4.$
45. $|x + 3| < 7.$
46. $|3 - x| < 5.$
47. $|2 - x| > 1.$
48. $|x + 1| > 2.$
49. $|2x - 1| < 7.$
50. $|3x - 4| > 3.$
51. $|4x - 3| < 9.$
52. $\dfrac{x - 1}{x - 2} < 0.$
53. $\dfrac{x - 3}{x + 5} > 0.$
54. $\dfrac{x - 4}{x} < 0.$
55. $\dfrac{x + 3}{x} > 0.$
56. $\dfrac{2x}{x - 3} < 0.$
57. $\dfrac{3x + 4}{x - 1} > 0.$

Chapter Fourteen

THE BINOMIAL THEOREM AND MATHEMATICAL INDUCTION

102. Factorial *n*. The product of consecutive integers beginning with 1 occurs frequently in mathematics. A shorthand notation is used to stand for an expression of this kind. Thus the product of the positive integers from 1 to n, inclusive, is designated by the symbol $n!$. This is read "factorial n."

Illustrations:

$$1! = 1,\quad 2! = 1\cdot 2,\quad 3! = 1\cdot 2\cdot 3,\quad 4! = 1\cdot 2\cdot 3\cdot 4.$$

Example 1. Evaluate $8!/5!$.

Solution.

$$\frac{8!}{5!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 6\cdot 7\cdot 8 = 336.$$

Example 2. Simplify the expression $(n + 1)!/(n - 1)!$.

Solution. The numerator has the integers from 1 to $n + 1$ as factors. The denominator has two fewer factors since it is the product of the integers from 1 to $n - 1$. We simplify the fraction by reducing it to lowest terms. Hence

$$\frac{(n+1)!}{(n-1)!} = \frac{1\cdot 2\cdot 3 \cdots (n-2)(n-1)(n)(n+1)}{1\cdot 2\cdot 3 \cdots (n-2)(n-1)} = n(n+1).$$

Exercise 70

Evaluate the following expressions:

1. $6!$.
2. $7!$.
3. $5!$.
4. $2!4!$.
5. $4! + 5!$.
6. $4!5!$.
7. $6! - 4! + 3!$.
8. $2(5!) - 5(2!)$.
9. $5(4!) - 4(4!)$.

10. $\frac{8!}{7!}$.
11. $\frac{10!}{8!}$.
12. $\frac{7!}{4!}$.
13. $\frac{5!}{8!}$.
14. $\frac{3!7!}{5!}$.
15. $\frac{12!}{6!6!}$.
16. $\frac{n!}{(n-1)!}$.
17. $\frac{(n+1)!}{n!}$.
18. $\frac{(n+2)!}{(n-1)!}$.
19. $(n-1)!n$.
20. $(n-2)!(n)(n-1)$.
21. $(n+1)(n!)$.

103. The Binomial Theorem. In Chapter One we saw how to write the square and the cube of a binomial. We wish now to show how to write any positive integral power of a binomial as a polynomial.

We may show by multiplying that

$$(a+x)^1 = a + x,$$

$$(a+x)^2 = a^2 + 2ax + x^2,$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4,$$

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

The right members of these identities are the expanded forms of the positive integral powers of $a + x$ from the first through the fifth. In each of the expansions we may observe the following facts:

1. The number of terms in each expansion is one more than the exponent of the binomial.
2. The first term is a with an exponent the same as the exponent of the binomial, and the exponent decreases by 1 from term to term.
3. The exponent of x in the second term is 1, and it increases by 1 from term to term.
4. The sum of the exponents of a and x in any term is equal to the exponent of the binomial.
5. The coefficients of the terms read in order from the first term through the last term are just the same as if read in the reverse order.
6. The coefficient of the second term is the same as the exponent of the binomial. The coefficient of any term farther on may be computed from the previous term by multiplying that term's coefficient by the exponent of a and dividing by 1 more than the exponent of x.

These statements are immediately evident except for item 6, which we notice further. The second term in the expansion of $(a+x)^4$ is $4a^3x$. We get the coefficient of the following term by multiplying 4 by 3 and dividing by 1 more than the exponent of x. Thus $(4\cdot3) \div 2 = 6$. Hence 6 is the coefficient of the next term. In this manner the coefficients can be computed from term to term.

In Section 107 we shall prove that the foregoing observations hold for the expansion of $(a + x)^n$, where n is any positive integer. This is called the **binomial theorem.** Assuming now that the theorem is true, we obtain

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}x^4 + \cdots + x^n.$$

This expansion is called the **binomial formula.** By use of it a positive integral power of a binomial can be conveniently written. The terms a and x may stand for terms of more than one factor or even stand for expressions of more than one term.

Example 1. Expand $(2 + x)^6$ and simplify.

Solution. The expansion can be found by use of the formula or by following the observations 1 through 6. We first apply the formula using $a = 2$ and $n = 6$. Thus we get

$$(2 + x)^6 = 2^6 + 6(2)^5x + \frac{6\cdot5}{2!}(2)^4x^2 + \frac{6\cdot5\cdot4}{3!}(2)^3x^3 + \frac{6\cdot5\cdot4\cdot3}{4!}(2)^2x^4$$
$$+ \frac{6\cdot5\cdot4\cdot3\cdot2}{5!}(2)x^5 + x^6$$
$$= 64 + 6(32)x + 15(16)x^2 + 20(8)x^3 + 15(4)x^4 + 6(2)x^5 + x^6$$
$$= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6.$$

Alternatively, we can use item 6 and compute the coefficients mentally in going from term to term. Thus we have

$$(2 + x)^6 = 2^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6.$$

The coefficient of the third term is obtained from the second term $6(2)^5x$ by computing $(6\cdot5) \div 2 = 15$. Similarly, the coefficient of the fourth term, using the third term, is $(15\cdot4) \div 3 = 20$. A continuation of the process yields the remaining coefficients. However, the coefficients after the fourth term are, in reverse order, exactly those of the first four terms. This expansion when simplified gives the same result as before.

Example 2. Expand $(2a^2 + 3y)^4$.
Solution.

$$(2a^2 + 3y)^4 = (2a^2)^4 + 4(2a^2)^3(3y) + 6(2a^2)^2(3y)^2 + 4(2a^2)(3y)^3 + (3y)^4$$
$$= 16a^8 + 96a^6y + 216a^4y^2 + 216a^2y^3 + 81y^4.$$

Example 3. Expand $(a - x)^6$ by the binomial theorem.
Solution.

$$(a - x)^6 = [a + (-x)]^6$$
$$= a^6 + 6a^5(-x) + 15a^4(-x)^2 + 20a^3(-x)^3 + 15a^2(-x)^4$$
$$+ 6a(-x)^5 + (-x)^6$$
$$= a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$$

The signs alternate here, as will be the case whenever the binomial is the difference of two quantities.

Exercise 71

Expand and then simplify:

1. $(a - x)^4$.
2. $(a + 2y)^4$.
3. $(a - b)^5$.
4. $(a + 2)^5$.
5. $(3a + 2y)^4$.
6. $(x + y)^6$.
7. $(x + 1)^7$.
8. $(1 - x)^8$.
9. $(1 + x)^7$.
10. $(m + n)^9$.
11. $(m - n)^{10}$.
12. $(m - n)^8$.
13. $(2a - y)^6$.
14. $(2x^2 - y)^5$.
15. $(a^2 + 2b)^5$.
16. $(2x - y^2)^7$.
17. $(a^2 - 3y)^6$.
18. $(x^3 - 2)^7$.
19. $\left(\frac{x}{2} - \frac{2}{x}\right)^6$.
20. $\left(\frac{a}{b} + \frac{b}{a}\right)^6$.
21. $\left(x + \frac{1}{x}\right)^6$.
22. $(x - y + z)^3$.
23. $(2a + b - 3c)^3$.
24. $(x^2 - x + 1)^3$.

104. The General Term of a Binomial Expansion. Frequently one term only of a binomial expansion is needed. By examining the expansion of $(a + x)^n$ it may be seen how any term may be found independently of the preceding terms. The following observations will enable us to write the rth term, where r is any integer from 2 to $n + 1$, inclusive.

1. The exponent of x in each term is one less than the number of the term. Hence the exponent of x in the rth term is $r - 1$.

2. Since the sum of the exponents of a and x in any term is equal to n, the exponent of a in the rth term is $n - r + 1$.

3. The denominator of the coefficient of each term in which x appears is the factorial of the exponent of x in the term. Hence the denominator of the coefficient of the rth term is $(r - 1)!$.

4. The number of factors in the numerator of each coefficient, after the first, is equal to the exponent of x in the term. Then the numerator of the coefficient of the rth term has $r - 1$ factors. The first factor is n; the second, $n - 1$; the third, $n - 2$; the fourth, $n - 3$, and so on. Hence the $(r - 1)$st factor is $n - (r - 1) + 1 = n - r + 2$. From these observations we may write

$$r\text{th term} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}\,a^{n-r+1}x^{r-1}.$$

This formula will give any term after the first. Assigning r the value $n + 1$ yields the last term x^n. The formula is equal to zero when r is assigned an integral value greater than $n + 1$. Thus for $r = n + 2$ the factor $n - r + 2$ becomes $n - (n + 2) + 2 = 0$. This shows that, according to the binomial formula, the expansion terminates with the $(n + 1)$st term.

Example. Write the 7th term of $(2x - y)^{11}$.

Solution. The exponent of $-y$ is 6, and the exponent of $2x$ is 5. The denominator of the coefficient is 6!. The six factors of the numerator are 11, 10, 9, 8, 7, 6. Hence

$$7\text{th term} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{6!}(2x)^5(-y)^6 = 462(32x^5)y^6 = 14{,}784x^5y^6.$$

105. The Binomial Theorem for Fractional and Negative Exponents. If n is zero or a positive integer, the expansion of $(a + x)^n$ has $n + 1$ terms. But for other values of n, the expansion does not end. That this is true may be seen by noticing the formula for the rth term. The factor $n - r + 2$ cannot be zero for any positive integral value of r unless n is zero or a positive integral. We can, however, write as many terms as we like of a nonterminating expansion. There are mathematical situations in which the first few terms yield useful results, provided values of x are chosen between $-a$ and a. We shall not give a proof of this statement. The question is considered in calculus.

Example 1. Find the first four terms of the expansion of $(a + 2x)^{-1/2}$.

Solution.

$$\begin{aligned}(a+2x)^{-1/2} &= a^{-1/2} - \frac{1}{2}a^{-3/2}(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}a^{-5/2}(2x)^2 \\ &\quad + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}a^{-7/2}(2x)^3 + \cdots \\ &= a^{-1/2} - a^{-3/2}x + \tfrac{3}{2}a^{-5/2}x^2 - \tfrac{5}{2}a^{-7/2}x^3 + \cdots.\end{aligned}$$

Example 2. Find approximately $\sqrt{15}$.
Solution.

$$\sqrt{15} = (16-1)^{1/2} = 16^{1/2} + \tfrac{1}{2}(16)^{-1/2}(-1) - \tfrac{1}{8}(16)^{-3/2}(-1)^2 \cdots$$

$$= 4 - \tfrac{1}{8} - \tfrac{1}{512}$$

$$= 4 - .125 - .002$$

$$= 3.873.$$

This gives approximately the square root of 15. The result may be had to more decimal places by using more terms of the binomial expansion.

Exercise 72

Express the indicated term in simplified form:

1. $(a+x)^{11}$, 7th term.
2. $(a-x)^{12}$, 6th term.
3. $(x-\frac{1}{2}y)^{10}$, 5th term.
4. $(2x-y)^8$, 5th term.
5. $(a^2-b)^{12}$, 9th term.
6. $(a^2-b^3)^{11}$, 8th term.
7. $(\frac{3}{4}a-\frac{4}{3}b)^8$, middle term.
8. $(2x^2-\frac{1}{5}y)^{13}$, 5th term.
9. $\left(x+\frac{1}{x}\right)^{12}$, term free of x.
10. $\left(x^2-\frac{1}{3x}\right)^{12}$, term free of x.
11. $\left(x^{1/2}+\frac{1}{2x}\right)^{9}$, term free of x
12. $\left(a^{1/3}-\frac{1}{b}\right)^{14}$, 7th term.
13. $\left(\frac{3}{y}+\frac{y^2}{9}\right)^{15}$, term free of y.
14. $\left(x^2+\frac{1}{2x^3}\right)^{15}$, 7th term.

Expand to four terms and simplify:

15. $(x+y)^{-1}$.
16. $(1-x)^{-1}$.
17. $(a+x)^{-2}$.
18. $(a+2x)^{-3}$.
19. $(a+3)^{-4}$.
20. $(4-x)^{-5}$.
21. $(1+y)^{1/2}$.
22. $(1-y)^{-1/3}$.
23. $(3+y)^{3/5}$.
24. $(2-x)^{1/4}$.
25. $(5+b)^{-2/3}$.
26. $(1-3x)^{-3/4}$.

Approximate to three decimal places, using three terms of expansions:

27. $\sqrt{24} = (25-1)^{1/2}$.
28. $\sqrt[3]{28} = (27+1)^{1/3}$.
29. $\sqrt[4]{82}$.
30. $\sqrt[3]{26}$.
31. $\sqrt{0.98}$.
32. $\sqrt{1.05}$.
33. $26^{-1/2}$.
34. $28^{-1/3}$.
35. $33^{-1/5}$.

106. Mathematical Induction. Certain mathematical theorems and formulas can be proved by a process of reasoning called **mathematical induction.** Though the reasoning involved is quite simple, mathematical induction can frequently be applied with great effectiveness. The method may be best explained with illustrative examples.

Example 1. Prove that the sum of the first n positive odd integers is n^2. That is, show that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Proof: The proof has two separate parts.

Part I. For the formula to be true it must hold in particular for the smallest integer which may be assigned to n. The smallest integer for which the formula would have meaning is $n = 1$. Then checking the formula for $n = 1$ constitutes the first part of the proof. To illustrate the meaning of the formula we substitute also $n = 2$ and $n = 3$.

For $n = 1$, $\qquad 1 = 1^2 \quad \text{or} \quad 1 = 1.$

For $n = 2$, $\qquad 1 + 3 = 2^2 \quad \text{or} \quad 4 = 4.$

For $n = 3$, $\qquad 1 + 3 + 5 = 3^2 \quad \text{or} \quad 9 = 9.$

Thus it is seen that the formula is true for $n = 1$, $n = 2$, and $n = 3$. But this does not prove the formula for other values of n, and further checking would make the proof only for the particular values used. To complete the proof we proceed to the next part.

Part II. This part of the proof consists in showing that if the formula holds for any value of n, say $n = k$, then consequently it also holds for $n = k + 1$. That is, if k^2 is the sum of the first k positive odd integers, then it must be shown that $(k + 1)^2$ is the sum of the first $k + 1$ positive odd integers. Hence, substituting k for n, we *assume* that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2.$$

Now if we add $2k + 1$, the next consecutive odd integer, to both sides of the equation we have

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1).$$

The left member of this equation contains as terms the first $k + 1$ positive odd integers. The right member, when factored, is $(k + 1)^2$. Thus if the given formula is true for $n = k$, it is true for the next n, or for $n = k + 1$.

Taking the two parts, let us notice what conclusions can be made. By Part I, the formula holds for $n = 1$. Then, by Part II, the formula holds for n one greater, or for $n = 2$. Since the formula holds for $n = 2$, by Part II again, it holds for $n = 3$. Thus Part II may be applied repeatedly to show that the formula holds for $n = 2, 3, 4$, and so on, for n any positive integer.

It should be noticed that both parts of a proof by mathematical induction are essential. The failure of either part in a supposed formula means that the formula holds for no value of n or, at most, for only some values of n. For example, the relation

$$1 + 2 + 3 + \cdots + n = 2n - 1$$

holds for $n = 1$ and for $n = 2$. It does not hold for any other value of n; and hence Part II could not be established.

Let us next test the formula

$$5 + 10 + 15 + \cdots + 5n = \tfrac{5}{2}n(n + 1) - 1.$$

Assuming that the formula holds for $n = k$, we have

$$5 + 10 + 15 + \cdots + 5k = \tfrac{5}{2}k(k + 1) - 1.$$

Add $5k + 5$ to both sides of the equation:

$$\begin{aligned} 5 + 10 + 15 + \cdots + 5k + (5k + 5) &= \tfrac{5}{2}k(k + 1) - 1 + (5k + 5) \\ &= \tfrac{5}{2}(k + 1)(k + 2) - 1. \end{aligned}$$

The right member of this equation is just what we would get by substituting $n = k + 1$ in the given formula. Thus Part II of the proof is established. But if we try Part I by substituting $n = 1$ in the supposed formula, we have

$$5 \neq \tfrac{5}{2}(1)(2) - 1 = 5 - 1 = 4.$$

Part I fails; the relation as given holds for no value of n.

Example 2. Prove by mathematical induction that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$

Proof: We must establish the two parts of the proof.

Part I. If we try the formula for $n = 1$, we have

$$1^2 = \tfrac{1}{6}(1)(1 + 1)(2 + 1) = \tfrac{1}{6}(2)(3) = 1.$$

Thus we see that the formula holds for $n = 1$. While this constitutes the demonstration of Part I, a few other small values of n may be tried if one so desires.

Part II. We next assume that the formula is true for $n = k$. That is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \tfrac{1}{6}k(k + 1)(2k + 1).$$

Now add the square of $k + 1$ to both sides of the equation:

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k + 1)^2 &= \tfrac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \\ &= \tfrac{1}{6}(k + 1)(k + 2)(2k + 3). \end{aligned}$$

The left member of this equation is the sum of the squares of the first $k + 1$ positive integers. And the expression in the right member is exactly that which the given formula yields for $n = k + 1$. That is,

$$\begin{aligned}\tfrac{1}{6}n(n + 1)(2n + 1) &= \tfrac{1}{6}(k + 1)(k + 1 + 1)[2(k + 1) + 1]\\ &= \tfrac{1}{6}(k + 1)(k + 2)(2k + 3).\end{aligned}$$

Thus, if the given formula holds for a value of n, it holds for n one greater. Combining the two parts of the proof, the formula holds for $n = 1$; and hence for $n = 2$. By repeated applications of Part II, the formula is seen to be true for all positive integral values of n.

Exercise 73

Prove that the following formulas are true for all positive integral values of n. Use mathematical induction.

1. $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.
2. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.
3. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1)$.
4. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$.
5. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$.
6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$.
7. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2}{3}n(n + 1)(2n + 1)$.
8. $1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1)$.
9. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n + 1)^2$.
10. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.
11. $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$.
12. $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{1}{2}n[2a + (n - 1)d]$.
13. $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$.
14. If n is a positive integer prove that (a) $x^n - y^n$ is divisible by $x - y$; (b) $x^{2n+1} + y^{2n+1}$ is divisible by $x + y$.
Hint: $x^n - y^n = x^n - xy^{n-1} + xy^{n-1} - y^n = x(x^{n-1} - y^{n-1}) + y^{n-1}(x - y)$.

107. Proof of the Binomial Formula for Positive Integral Exponents. We now use mathematical induction to prove that the binomial formula gives the expansion of $(a + x)^n$, where n is any positive integer. Since expansions according to the formula have been tested for several small values of n, we proceed to the second part of the proof.

Part II. We assume that the formula is true for $n = k$. Then we have

$$(a + x)^k = a^k + ka^{k-1}x + \cdots + \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}a^{k-r+1}x^{r-1}$$
$$+ \frac{k(k-1)\cdots(k-r+1)(k-r+1)}{r!}a^{k-r}x^r + \cdots + kax^{k-1} + x^k. \quad (1)$$

To obtain an expression for $(a + x)^{k+1}$, we multiply both members of equation (1) by $a + x$. This gives

$$(a + x)^{k+1} = a^{k+1} + (k+1)a^kx + \cdots$$
$$+ \frac{(k+1)(k)(k-1)\cdots(k-r+2)}{r!}a^{k-r+1}x^r + \cdots + x^{k+1}.$$

The student should verify this multiplication. It is easy to check the first, second, and last terms. The term containing $a^{k-r+1}x^r$ is obtained by multiplying the rth term of the expansion in (1) by x and the $(r + 1)$st term by a and then adding these two products. Thus the coefficient of $a^{k-r+1}x^r$ is the sum of the coefficients of the rth term and the $(r + 1)$st term of the right member of (1). That is,

$$\frac{k(k-1)\cdots(k-r+2)}{(r-1)!} + \frac{k(k-1)\cdots(k-r+2)(k-r+1)}{r!}$$
$$= \frac{rk(k-1)\cdots(k-r+2) + k(k-1)\cdots(k-r+2)(k-r+1)}{r!}$$
$$= \frac{k(k-1)\cdots(k-r+2)[r + (k-r+1)]}{r!}$$
$$= \frac{(k+1)(k)(k-1)\cdots(k-r+2)}{r!}.$$

The expression which we have obtained for $(a + x)^{k+1}$ is exactly what the binomial formula yields by substituting $n = k + 1$. Hence for n any positive integer we have

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1} + \cdots + x^n.$$

Supplementary Problems

Expand and then simplify:

1. $(x^{-1} + 2)^3$.
2. $(x^{1/3} + y^{1/3})^3$.
3. $(a^{-2} - b)^4$.
4. $(xy^2 - 2z^3)^5$.
5. $(1 + 3\sqrt{x})^5$.
6. $(\frac{1}{2}x - \sqrt{3})^6$.
7. $(x^{1/2} - \frac{1}{2}y)^6$.
8. $(\sqrt{a} - \sqrt{b})^5$.
9. $(\sqrt{a} + \sqrt{b})^7$.

The coefficients of the expansions of nonnegative integral powers of a binomial can be arranged in a triangular form known as **Pascal's triangle.** The coefficients of the expansions of each of the indicated powers of $a + x$ are written to the right.

$$\begin{array}{lccccccccccccc}
(a+x)^0 & & & & & & & 1 & & & & & & \\
(a+x)^1 & & & & & & 1 & & 1 & & & & & \\
(a+x)^2 & & & & & 1 & & 2 & & 1 & & & & \\
(a+x)^3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
(a+x)^4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
(a+x)^5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
(a+x)^6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}$$

The first and the last number of each row is 1. Each of the other numbers is equal to the sum of the two nearest numbers in the row just above it.

10. Form a Pascal triangle of ten rows. Use the numbers of the tenth row to write the expansion of $(a + x)^9$.

11. Find the value of $(1.01)^{10} = (1 + 0.01)^{10}$ correct to three decimal places by using three terms of the expansion.

12. Find the value of $(0.98)^5 = (1 - 0.02)^5$ correct to three decimal places. Use three terms of the expansion.

13. Compute $(1.2)^5$ correct to one decimal place.

14. Compute $(0.97)^5$ correct to two decimal places.

15. Compute $(1.03)^{10}$ to two decimal places.

Use the binomial formula to find the values of the indicated powers:

16. $98^2 = (100 - 2)^2$.
17. $99^3 = (100 - 1)^3$.
18. $81^2 = (80 + 1)^2$.
19. 55^2.
20. 92^2.
21. 161^2.

Chapter Fifteen

THEORY OF EQUATIONS

108. Rational Integral Equations. An equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

where n is a positive integer and the a's are constants, is called a **rational integral equation** in one unknown. If we assume that $a \neq 0$, the equation is of the nth degree. Since the left member is a polynomial, the equation is also called a polynomial equation. Using the notation of Section 39, we may let $f(x)$ stand for the polynomial. The equation then can be written more briefly as $f(x) = 0$.

Our chief interest in polynomial equations will be that of finding the roots. We have developed methods by which linear and quadratic equations can be solved quite readily. Finding the roots of higher degree equations is, in general, less simple. In fact, the problem is much too difficult for a complete investigation at this stage. We shall, however, consider certain propperties of polynomial equations and describe some methods of solving. We initiate our study by establishing some useful theorems.

109. The Remainder Theorem. If we divide $f(x) = 2x^3 - 3x^2 + x - 1$ by $x - 2$, we get the quotient $2x^2 + x + 3$ with 5 as the remainder. Now by substituting 2 for x in the dividend, we have $f(2) = 2(2)^3 - 3(2)^2 + 2 - 1 = 5$. Thus we obtain the remainder 5 in two ways. This is an illustration of a theorem which, as we shall see, has important consequences.

The Remainder Theorem. *If a polynomial $f(x)$ is divided by $x - r$ until a remainder free of x is obtained, this remainder is equal to $f(r)$.*

Proof: Let $Q(x)$ stand for the quotient obtained when $f(x)$ is divided by $x - r$, and let R stand for the remainder. Then

$$f(x) = (x - r)Q(x) + R.$$

(Dividend = divisor × quotient + remainder.)

Since this is an identity, true for all values of x, we may substitute r for x, and obtain

$$f(r) = (r - r)Q(r) + R$$

$$f(r) = 0 \cdot Q(r) + R$$

$$\boldsymbol{f(r) = R.}$$

This establishes the remainder theorem.

110. The Factor Theorem and Its Converse. *If r is a root of the rational integral equation $f(x) = 0$, then $x - r$ is a factor of $f(x)$; and conversely, if $x - r$ is a factor of $f(x)$, then r is a root of $f(x) = 0$.*

Proof: From the remainder theorem, we have

$$f(x) = (x - r)Q(x) + R = (x - r)Q(x) + f(r).$$

If r is a root of the equation $f(x) = 0$, then $f(r) = 0$. Hence we may present $f(x)$ in the factored form

$$f(x) = (x - r)Q(x).$$

To prove the converse of the theorem we need but to notice that with $x - r$ given as a factor, we may write the identity

$$f(x) = (x - r)Q(x),$$

which is true for all the values of x. Whence, replacing x by r, we have

$$f(r) = (r - r)Q(r) = 0 \cdot Q(r) = 0.$$

Example 1. Show by the factor theorem that $x^3 + x^2 - 7x - 3$ is divisible by $x + 3$.

Solution. To determine if $x + 3$ is a factor we need only to see if -3 is a root of the equation

$$x^3 + x^2 - 7x - 3 = 0.$$

Substituting -3 for x, we obtain

$$(-3)^3 + (-3)^2 - 7(-3) - 3 = -27 + 9 + 21 - 3 = 0.$$

The equation has -3 as a root; hence $x + 3$ is a factor of $x^3 + x^2 - 7x - 3$.

Example 2. Form a cubic equation $f(x) = 0$ whose roots are $2, -3, \frac{1}{5}$.

Solution. From the factor theorem we know that $x - 2$, $x + 3$, and $x - \frac{1}{5}$ are factors of $f(x)$. Hence we write the equation

$$(x - 2)(x + 3)(x - \tfrac{1}{5}) = 0.$$

This equation has the required roots. To avoid fractional coefficients, however, we replace $x - \frac{1}{5}$ by $5x - 1$ and obtain an equivalent equation. That is,

$$(x - 2)(x + 3)(5x - 1) = 0 \quad \text{or} \quad 5x^3 + 4x^2 - 31x + 6 = 0.$$

Example 3. Determine by the factor theorem if $x - y$ and $x + y$ are factors of $x^{2n} - y^{2n}$, where n is a positive integer.

Solution. We have $f(x) = x^{2n} - y^{2n}$ and $x - r = x - y$. Then

$$f(y) = y^{2n} - y^{2n} = 0.$$

Since y is a root of $x^{2n} - y^{2n} = 0$, $x - y$ is a factor of $x^{2n} - y^{2n}$. Similarly, $f(-y) = (-y)^{2n} - y^{2n} = 0$, and $x + y$ is a factor of $x^{2n} - y^{2n}$.

Exercise 74

Find the remainder when the polynomial is divided by the linear expression. Use both long division and the remainder theorem.

1. $x^3 - 4x^2 - 2x + 5$, $x - 1$.
2. $3x^3 + 4x^2 - 5x - 3$, $x + 1$.
3. $2x^3 + 5x^2 + x - 7$, $x - 2$.
4. $x^4 - 3x^3 - 2x^2 + 5x - 6$, $x - 3$.

Using the remainder theorem, find what the remainder would be if the first expression were divided by the second.

5. $2x^3 - x^2 + 3x + 1$, $x + 2$.
6. $x^3 - 5x^2 - 6x - 7$, $x - 2$.
7. $x^3 - 3x^2 - 2x - 2$, $x + 3$.
8. $4x^3 + 4x^2 + x + 3$, $x + 4$.
9. $2x^3 - 5x^2 - 8x + 3$, $x - \frac{1}{2}$.
10. $6x^3 + 9x^2 - 5x - 2$, $x - \frac{1}{3}$.
11. $x^4 - x^3 - x^2 + x - 1$, $x - 3$.
12. $x^4 + 2x^3 - 3x - 5$, $x + 2$.
13. $x^5 + 5x^4 + x^2 - 9$, $x + 2$.
14. $4x^4 + 2x^3 + 8x - 1$, $x + \frac{1}{2}$.

By means of the factor theorem, show that the second expression is a factor of the first.

15. $x^3 - 4x^2 + 3x + 8$, $x + 1$.
16. $2x^3 - 3x^2 + 2x - 8$, $x - 2$.
17. $3x^3 + 5x^2 - 6x + 18$, $x + 3$.
18. $4x^3 - 6x^2 + 2x - 12$, $x - 2$.
19. $x^3 + 2x^2 - 6x - 72$, $x - 4$.
20. $x^3 + 8x^2 + 8x - 32$, $x + 4$.
21. $2x^4 + x^3 + x - \frac{3}{4}$, $x - \frac{1}{2}$.
22. $3x^4 - x^3 - 3x^2 + \frac{1}{3}$, $x - \frac{1}{3}$.

23. $x^5 - 1$, $x - 1$.
24. $x^5 + 1$, $x + 1$.
25. $x^8 - 1$, $x + 1$.
26. $x^7 - y^7$, $x - y$.
27. $x^9 + y^9$, $x + y$.
28. $x^6 - y^6$, $x + y$.

29. $x^n - y^n$, $x - y$, n a positive integer.
30. $x^n - y^n$, $x + y$, n an even positive integer.
31. $x^n + y^n$, $x + y$, n an odd positive integer.

Write equations with integral coefficients which have the following numbers, and no others, as roots:

32. $1, 2, -3$.
33. $1, 2, 4$.
34. $-1, -2, -3$.
35. $2, -2, 3$.
36. $0, 1, 4$.
37. $-4, 3, 0$.
38. $\frac{1}{2}, 1, 2$.
39. $\frac{1}{3}, -1, 1$.
40. $\frac{1}{3}, \frac{2}{3}, -1$.
41. $-\sqrt{2}, \sqrt{2}, 2$.
42. $1 + \sqrt{2}, 1 - \sqrt{2}$.
43. $0, -\sqrt{3}, \sqrt{3}$.

111. Synthetic Division. A much used operation in finding roots of equations is that of dividing a polynomial $f(x)$ by a linear expression of the form $x - r$. This kind of division can be done rapidly by a process called **synthetic division.** The process is the result of eliminating nonessential writing from the method of long division. An explanation can best be made by an example.

Example 1. Divide $2x^3 + x^2 - 18x + 7$ by $x - 2$.
Solution. We first use the long division method.

$$\begin{array}{r|l}
2x^3 + x^2 - 18x + 7 & x - 2 \quad \text{(Divisor)} \\
2x^3 - 4x^2 \phantom{{} - 18x + 7} & \overline{2x^2 + 5x - 8} \quad \text{(Quotient)} \\
\cline{1-1}
5x^2 - 18x \phantom{{} + 7} & \\
5x^2 - 10x \phantom{{} + 7} & \\
\cline{1-1}
-8x + 7 & \\
-8x + 16 & \\
\cline{1-1}
-9 \quad \text{(Remainder)} &
\end{array}$$

In the steps of the division the terms $-18x$ and 7 were rewritten. Now let us leave these terms in the first line and write beneath them the terms to be subtracted. The division then appears as

$$\begin{array}{r|l}
2x^3 + x^2 - 18x + 7 & x - 2 \\
2x^3 - 4x^2 - 10x + 16 & \overline{2x^2 + 5x - 8} \\
\cline{1-1}
5x^2 - 8x - 9 &
\end{array}$$

The third line has in order the coefficients 5, -8, and -9. Placing the first coefficient of the first line before these, we have

$$2, 5, -8, -9.$$

These values, in proper order, are the coefficients of the quotient $2x^2 + 5x - 8$ and the remainder -9. Thus the coefficient of x^3 and the coefficients of the third line are the essential numbers to yield the quotient and the remainder. Let us now observe, step by step, how the third line is formed. The term $2x^3$ of the second line is the same as the first term of the first line. Its coefficient multiplied by -2 of the divisor gives -4, the coefficient of the second term of the second line. By subtracting $-4x^2$ from the term x^2 above, we get $5x^2$ as the first term of the third line. Its coefficient times -2 of the divisor is -10, the third coefficient of the second line. Subtraction then yields $-8x$, the second term of the third line. The product of -8 and -2 is 16, the fourth term of the second line. And the last subtraction gives the remainder -9. Thus the dividend and the

-2 of the divisor permit us to write the second and third lines, and thereby obtain the quotient and the remainder. Next we write only these parts and, in order to have all the coefficients in the third line, move $2x^3$ from the second line down to the third line. Then we have

$$\begin{array}{rrrr|l} 2x^3 + & x^2 - & 18x + & 7 & -2 \\ \cline{5-5} & -\ 4x^2 - & 10x + & 16 & \\ \hline 2x^3 + & 5x^2 - & 8x - & 9 & \end{array}$$

The powers of x are not essential in forming the coefficients of the second and third lines. Omitting them and writing only the coefficients, we obtain

$$\begin{array}{rrrr|l} 2 + & 1 - & 18 + & 7 & -2 \\ \cline{5-5} & -\ 4 - & 10 + & 16 & \\ \hline 2 + & 5 - & 8 - & 9 & \end{array}$$

The third line numbers, after the first, are obtained by subtraction. If we change the sign of -2 of the divisor, the signs of the numbers in the second line will be reversed. Then addition, rather than subtraction, completes the numbers of the third line. Making these changes in signs, we have in final form all the necessary work to give the quotient and remainder.

$$\begin{array}{rrrr|l} 2 + & 1 - & 18 + & 7 & 2 \\ \cline{5-5} & +\ 4 + & 10 - & 16 & \\ \hline 2 + & 5 - & 8 - & 9 & \end{array}$$

The quotient may be written, by supplying the proper powers of x, as

$$2x^2 + 5x - 8$$

with -9 as the remainder.

Following the process above, we may formulate directions for dividing a polynomial $f(x)$ by $x - r$.

1. Arrange the coefficients of $f(x)$ in a line in order of descending powers of x, supplying zero as the coefficient of each missing power of x. At the end of this line write r.

2. Rewrite the first coefficient of this line underneath as the first number of the third line. Multiply this number by r and place the product immediately beneath the second number of the first line, and write the sum of these two second position numbers as the second number of the third line. Continue this process until the second and third lines are completed to the position of the last number of the first line, which is the constant term of (x).

3. The numbers of the third line give the coefficients of the quotient in order of descending powers of x, the last number being the remainder. The degree of the quotient is, of course, one less than the degree of the dividend.

Example 2. Divide $3x^4 - 8x^3 - 7x - 8$ by $x - 3$.

Solution. We place zero in the first line as the coefficient of the missing power of x.

$$\begin{array}{r|l} 3 - 8 + 0 - 7 - 8 & 3 \\ +9 + 3 + 9 + 6 & \\ \hline 3 + 1 + 3 + 2 - 2 & \end{array}$$

The quotient is $3x^3 + x^2 + 3x + 2$ and the remainder is -2.

Example 3. Divide $2x^4 + 5x^3 - 8x^2 - 7x - 9$ by $x + 2$.

Solution. The divisor is $x - r = x - 2$ and therefore $r = -2$.

$$\begin{array}{r|l} 2 + 5 - \ \ 8 - \ \ 7 - \ \ 9 & -2 \\ -4 - \ \ 2 + 20 - 26 & \\ \hline 2 + 1 - 10 + 13 - 35 & \end{array}$$

The quotient is $2x^3 + x^2 - 10x + 13$ and the remainder is -35.

Recalling that when $f(x)$ is divided by $x - r$, the remainder is $f(x)$, we may use synthetic division to find the value of the polynomial corresponding to a given value of x.

Example 4. Find the value of $f(x) = 4x^3 - 7x^2 + 13x$ when x is replaced by -3.

Solution.

$$\begin{array}{r|l} 4 - \ \ 7 + 13 + \ \ 0 & -3 \\ -12 + 57 - 210 & \\ \hline 4 - 19 + 70 - 210 & \end{array}$$

The remainder is -210; hence $f(-3) = -210$.

Exercise 75

Find the quotient and remainder in each problem by the use of synthetic division.

1. $(2x^3 - 7x^2 + 3x + 1) \div (x - 1)$.
2. $(x^3 + 5x^2 - 5x - 6) \div (x + 1)$.
3. $(x^3 - x - 9x^2 + 11) \div (x - 2)$.
4. $(3x^3 + 3x^2 + 5x + 1) \div (x + 2)$.
5. $(4x^3 + 4x^2 + x + 5) \div (x + 3)$.
6. $(2x^3 - 8x^2 + 5x + 3) \div (x - 3)$.
7. $(5x^3 + 8x^2 + 7x + 4) \div (x + 1)$.
8. $(4x^3 - 4x^2 + 3x - 6) \div (x - 1)$.
9. $(x^4 - x^3 + 3x + 1) \div (x + 1)$.
10. $(x^4 + x^2 - 5x - 5) \div (x - 1)$.
11. $(2x^4 + 7x^3 + x + 13) \div (x + 3)$.
12. $(x^4 - x^3 - 7x^2 - 3) \div (x - 4)$.
13. $(x^5 - x^4 - x^2 + 1) \div (x - 2)$.
14. $(x^5 + x^3 + x + 1) \div (x + 2)$.

15. $(x^5 + 2x^4 - 5x^3 + 4x^2 - 5x + 3) \div (x + 4)$.
16. $(x^5 - 4x^4 - 6x^3 + 5x^2 - 6x + 7) \div (x - 5)$.
17. $(2x^5 - 3x^4 - x^3 + x^2 + 4x + 4) \div (x - \frac{1}{2})$.

Evaluate by synthetic division:

18. If $f(x) = x^3 - 7x^2 + 12x - 2$, find $f(-2)$; $f(3)$; $f(4)$.
19. If $f(x) = x^4 - 7x^3 + 7x - 20$, find $f(1)$; $f(-1)$; $f(2)$.
20. If $f(x) = 4x^3 + 7x^2 - 13x + 20$, find $f(2)$; $f(3)$; $f(-3)$.
21. If $f(x) = 2x^4 + 6x^3 + 7x^2 - x + 7$, find $f(-2)$; $f(-1)$; $f(1)$.
22. If $f(x) = x^3 - 4x^2 - 20x + 50$, find $f(5)$; $f(6)$; $f(7)$.

112. The Graph of a Polynomial. In Section 41 graphs of functions were introduced, and in Section 82 the real roots of quadratic equations were found by graphing the corresponding quadratic functions. The same methods of graphing are to be used here, but with the added advantage of computing the functional values by synthetic division. Moreover, the real zeros of $f(x)$, or the real roots of $f(x) = 0$, may be had by noticing the abscissas of the points where the graph meets the x-axis.

Example. Solve the equation $x^3 - 4x^2 + x + 3 = 0$ graphically for the real roots, estimating them to one decimal place.

Solution. To draw the graph of the function $f(x) = x^3 - 4x^2 + x + 3$, we first set $y = f(x)$ and prepare a table of corresponding values of x and y. The functional values should be verified by synthetic division. When $x = 3.5$, for example, we have

$$\begin{array}{rrrr|l} 1 & -\ 4.0 & +\ 1.00 & +\ 3.000 & 3.5 \\ & +\ 3.5 & -\ 1.75 & -\ 2.625 & \\ \hline 1 & -\ 0.5 & -\ 0.75 & +\ 0.375 & \end{array}$$

x	y
−1	−3
−0.5	1.4
0	3
1	1
1.5	−1.1
2	−3
3	−3
3.5	0.4
4	7

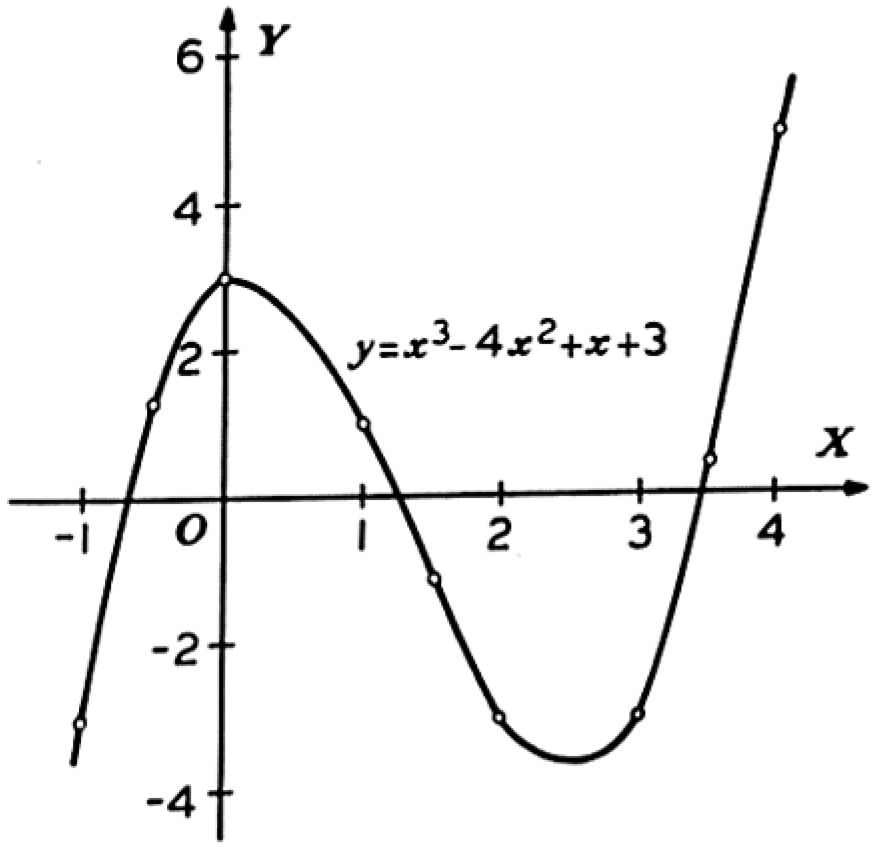


FIG. 21.

The graph of $f(x)$ is shown in Figure 21. From the graph the zeros of the function, or the roots of the given equation, appear to be approximately -0.7, 1.2, and 3.5.

Exercise 76

Solve for the real roots by graphing. Estimate the results to one decimal place.

1. $x^3 - 4x^2 + x + 5 = 0$.
2. $x^3 + 4x^2 + x - 4 = 0$.
3. $x^3 - 3x + 4 = 0$.
4. $x^3 - 7x - 5 = 0$.
5. $x^3 + x^2 - 9x - 7 = 0$.
6. $x^3 - 2x^2 - 4x + 6 = 0$.
7. $x^3 - x^2 - 5x - 1 = 0$.
8. $x^3 - 19x + 25 = 0$.
9. $x^3 - 3x^2 + 3 = 0$.
10. $x^3 + 3x^2 + 1 = 0$.
11. $x^3 - x^2 - 5x - 6 = 0$.
12. $x^3 - 3x^2 - 7x - 6 = 0$.
13. $x^4 - 2x^2 - 3 = 0$.
14. $x^4 - 4x^2 + 3 = 0$.
15. $x^4 - 3x + 5 = 0$.
16. $x^4 + 4x^3 - 16x + 5 = 0$.

113. Theorems Concerning Roots. Having found roots of certain rational integral equations, particularly equations of the first and second degrees, we might expect that every equation of this kind has a root. In this connection we state the following theorem.

THEOREM 1. *Every rational integral equation in one unknown has at least one root, real or imaginary.*

This important theorem is known as the **fundamental theorem of algebra.** The proof, which we omit because of its difficulty, was first made in 1797 by a German mathematician named Gauss. We shall assume this theorem to be true and from it establish other theorems.

THEOREM 2. *Every polynomial $f(x)$ of degree n can be expressed as the product of n linear factors.*

Proof: Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots a_{n-1}x + a_n$, with $a_0 \neq 0$. From the fundamental theorem, $f(x) = 0$ has at least one root. Denoting this root by r_1 and recalling the factor theorem, we conclude that $x - r_1$ is a factor of $f(x)$. Hence,

$$f(x) = (x - r_1)Q_1(x),$$

where the factor $Q_1(x)$ is a polynomial of degree $n - 1$, its highest degree term being a_0x^{n-1}. Using the fundamental theorem and the factor theorem again, we have

$$Q_1(x) = (x - r_2)Q_2(x).$$

Whence

$$f(x) = (x - r_1)(x - r_2)Q_2(x),$$

where r_2 is a root of $Q_1(x) = 0$, and $Q_2(x)$ is a polynomial with a_0x^{n-2} its highest degree term. This process may be continued until n linear factors are obtained, the last factor having the term a_0x. Then factoring the constant a_0 from the last factor we may write it as $a_0(x - r_n)$ and have

$$f(x) = a_0(x - r_1)(x - r_2)\cdots(x - r_n).$$

The numbers $r_1, r_2, \cdots, r_n$ need not all be distinct, or even real. That is, certain factors may be repeated and some, or all, may contain imaginary constants.

THEOREM 3. *Every rational integral equation $f(x) = 0$ of degree n has exactly n roots.*

Proof: By the preceding theorem we may factor $f(x)$ and write the equation as

$$a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0.$$

The roots of the equation are $r_1, r_2, \cdots, r_n$ because any one of these values when substituted for x makes a factor equal to zero, and hence the product of the factors equals zero. A value of x different from these quantities will obviously make no factor equal to zero. Hence the equation has only the n roots $r_1, r_2, \cdots, r_n$.

If no two factors of $f(x)$ are alike, the n roots are all different. Each root is then called a **single,** or **simple, root.** If the same factor occurs twice, a root occurs twice. A root which occurs twice is called a **double root.** If a factor occurs three times, there is a **triple root.** In general, if a factor occurs m times, the corresponding root is said to be of **multiplicity m.** It is customary to count a double root as two roots, a triple root as three roots, and a root of multiplicity m as m roots. Counting the roots in this way, we say that a rational integral equation of degree n has exactly n roots.

Illustrations: The equation $(x - 1)(x + 2)(x - 5) = 0$ is of degree three. It has the three distinct roots 1, -2, and 5. The equation

$$(x - 4)^2(x - 3)^4(x + 2) = 0$$

is of degree seven. It has 4 as a double root, 3 as a root of multiplicity four, and -2 as a simple root. The total number of roots, giving each a count according to its multiplicity, is seven.

THEOREM 4. *If two polynomials of degree n are equal for more than n distinct values of the variable, they are identical and consequently are equal for all values of the variable.*

Proof: Denoting the polynomials by $f(x)$ and $g(x)$, where

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n,$$

and

$$g(x) = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n,$$

we have

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$
$$= b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n.$$

By transposing, we obtain

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + (a_2 - b_2)x^{n-2} + \cdots$$
$$+ (a_{n-1} - b_{n-1})x + a_n - b_n = 0,$$

which is an equation of degree not greater than n; and therefore can have no more than n roots. Then the only way in which more than n distinct values of x can satisfy the equation is for all the coefficients to be zero. That is,

$$a_0 - b_0 = 0, \quad a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \quad \cdots, \quad a_{n-1} - b_{n-1} = 0,$$
$$a_n - b_n = 0$$

or

$$a_0 = b_0, \quad a_1 = b_1, \quad a_2 = b_2, \quad \cdots, \quad a_{n-1} = b_{n-1}, \quad a_n = b_n.$$

The polynomials $f(x)$ and $g(x)$, having equal coefficients for like powers of x, are identical and consequently are equal for all values of x.

114. Imaginary and Quadratic Surd Roots. From the quadratic formula it may be seen that if the imaginary number $(a + bi)$ is a root of a quadratic equation with real coefficients, then the other root is $(a - bi)$. That is, the imaginary roots occur in pairs, which are called conjugate imaginary numbers. We now prove a general theorem concerning imaginary roots.

THEOREM 1. *If the imaginary number $(a + bi)$ is a root of a rational integral equation with real coefficients, then the imaginary number $(a - bi)$ is also a root.*

Proof: Designating the equation by $f(x) = 0$, we have $f(a + bi) = 0$ since $(a + bi)$ is a root of the equation. Then $(x - a - bi)$ is a factor of $f(x)$. We shall show that $(x - a + bi)$ is also a factor by showing that the product

$$(x - a - bi)(x - a + bi) = x^2 - 2ax + (a^2 + b^2)$$

is a factor of $f(x)$. Let $D(x)$ stand for this product. Now $f(x)$ can be divided by the quadratic expression $D(x)$ until a remainder of degree lower than two is obtained. Then indicating the quotient by $Q(x)$ and the remainder by $Rx + S$, we may write

$$f(x) = D(x)Q(x) + Rx + S.$$

If we substitute $a + bi$ for x in this identity, we get

$$f(a + bi) = D(a + bi)Q(a + bi) + R(a + bi) + S.$$

But, by hypothesis, $f(a + bi) = 0$. And $D(a + bi) = 0$ since $(x - a - bi)$ is a factor of $D(x)$. Thus we obtain

$$0 = 0 \cdot Q(a + bi) + R(a + bi) + S,$$

or

$$(Ra + S) + Rbi = 0.$$

A complex number is equal to zero if, and only if, the real and imaginary parts are each equal to zero. Hence we must have

$$Ra + S = 0 \quad \text{and} \quad Rb = 0.$$

Since $a + bi$ is an imaginary number, $b \neq 0$; and therefore $R = 0$. Substituting $R = 0$ in $Ra + S = 0$ gives $S = 0$. Hence the remainder $Rx + S$ is equal to zero, and

$$f(x) = D(x)Q(x) = (x - a - bi)(x - a + bi)Q(x).$$

Then $(x - a + bi)$ being a factor of $f(x)$, it follows that $(a - bi)$ is a root of $f(x) = 0$.

If we restrict the coefficients of $f(x) = 0$ to be rational numbers, then we have a similar theorem concerning quadratic surd roots. The number $a + \sqrt{b}$ is a quadratic surd if a and b are rational numbers (b positive) and if $\sqrt{b}$ is irrational.

THEOREM 2. *If the quadratic surd $(a + \sqrt{b}\,)$ is a root of a rational integral equation with rational coefficients, then the conjugate surd $(a - \sqrt{b}\,)$ is also a root.*

The proof of this theorem is much like that for the conjugate imaginary roots, and will not be given.

Example 1. Solve the equation $x^4 - 4x^3 + 10x^2 + 12x - 39 = 0$, if $(2 - 3i)$ is one of the roots.

Solution. Since $(2 - 3i)$ is a root, $(2 + 3i)$ is also a root. Then the product

$$(x - 2 + 3i)(x - 2 - 3i) = x^2 - 4x + 13$$

is a factor of the left member of the given equation. The other factor, obtained by division, is $x^2 - 3$. The roots of $x^2 - 3 = 0$ are $\pm\sqrt{3}$. Hence the solution of the given equation is

$$x = \pm\sqrt{3}, \quad 2 \pm 3i.$$

Example 2. Form an equation with rational coefficients and of the lowest possible degree if two of the roots are $2 - 3\sqrt{5}$ and $4 + i$.

Solution. In addition to the given roots, $2 + 3\sqrt{5}$ and $4 - i$ must also be roots. Hence the required equation is

$$(x - 2 + 3\sqrt{5})(x - 2 - 3\sqrt{5})(x - 4 - i)(x - 4 + i) = 0,$$

or, multiplying,

$$x^4 - 12x^3 + 8x^2 + 260x - 697 = 0.$$

115. Coefficients in Terms of Roots. If r_1, r_2, and r_3 are the roots of the cubic equation

$$x^3 + a_1x^2 + a_2x + a_3 = 0, \tag{1}$$

then the equation may be written in the factored form

$$(x - r_1)(x - r_2)(x - r_3) = 0. \tag{2}$$

By multiplication we get

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0. \tag{3}$$

The left members of equations (1) and (3) are identically equal; hence we may equate the coefficients of the like powers of x and obtain

$$a_1 = -(r_1 + r_2 + r_3),$$

$$a_2 = (r_1r_2 + r_1r_3 + r_2r_3),$$

$$a_3 = -(r_1r_2r_3).$$

In the same manner it could be shown that the coefficients $a_1, a_2, \cdots, a_n$ of the equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{4}$$

may be expressed as follows:

$a_1 = -$(*the sum of the roots*),
$a_2 =$ (*the sum of the products of the roots taken two at a time*),
$a_3 = -$(*the sum of the products of the roots taken three at a time*),
. . . .
$a_n = (-1)^n$ (*product of the roots*).

If the coefficient of the highest degree term is a_0, different from 1, then the equation can be put in the form (4) by dividing by a_0.

Example. Form a cubic equation which has 2, -3, and 5 as roots.

Solution. Write the equation as $x^3 + a_1x^2 + a_2x + a_3 = 0$. Then

$$a_1 = -(2 - 3 + 5) = -(4) = -4,$$

$$a_2 = (2)(-3) + (2)(5) + (-3)(5) = -6 + 10 - 15 = -11,$$

$$a_3 = -(2)(-3)(5) = 30.$$

Hence we obtain the equation $x^3 - 4x^2 - 11x + 30 = 0$, which has the given numbers as roots.

Exercise 77

Find the roots which are not given:

1. $x^3 + 4x^2 + x + 4 = 0$. i is a root.
2. $x^3 - 3x^2 + 9x + 13 = 0$. $2 - 3i$ is a root.
3. $x^3 - 8x^2 + 6x + 52 = 0$. $5 - i$ is a root.
4. $2x^3 + 3x^2 - 10x - 15 = 0$. $\sqrt{5}$ is a root.
5. $3x^3 - 7x^2 - x + 1 = 0$. $1 - \sqrt{2}$ is a root.
6. $x^4 + 6x^3 + 17x^2 + 36x + 66 = 0$. $-3 + i\sqrt{2}$ is a root.
7. $x^4 - 2x^3 + 4x^2 + 2x - 5 = 0$. $1 - 2i$ is a root.
8. $x^4 - 4x^3 + x^2 + 28x - 56 = 0$. $\sqrt{7}$ is a root.
9. $x^4 - 2x^3 + 2x^2 + 4x - 8 = 0$. $1 + i\sqrt{3}$ is a root.
10. $x^4 + 4x^3 + 2x^2 + 4x + 1 = 0$. $-2 - \sqrt{3}$ is a root.

Find the equation of lowest degree with integral coefficients which has the given numbers among its roots:

11. $1 - i, -2$. **12.** $6 + 3i, 4$. **13.** $i, 2 - i$.
14. $5 + 2i, \sqrt{2}$. **15.** $3i, 4 - \sqrt{2}$. **16.** $1 + \sqrt{2}, 2 - \sqrt{3}$.
17. $\sqrt{7}, 3 - \sqrt{5}$. **18.** $1 - \sqrt{2}, 2 + i$. **19.** $2 - 4i, 4 - \sqrt{3}$.

Find the sum and the product of the roots of each of the following equations:

20. $x^3 + 4x^2 - 5x + 7 = 0$. **21.** $x^3 - 3x^2 + x - 5 = 0$.
22. $x^4 + x^2 - 5x + 1 = 0$. **23.** $x^4 - 7x^3 + 2x^2 + 3x = 0$.
24. $x^5 + 6x^4 + x - 10 = 0$. **25.** $x^6 - x^4 + x^2 + 3 = 0$.
26. $2x^3 - 5x^2 + 6x + 8 = 0$. **27.** $5x^4 + 10x^3 + x - 14 = 0$.

Use the method of Section 115 to find an equation having the given roots:

28. $-2, 1, 4$. **29.** $0, 3, -5$. **30.** $-3, -2, 1$.
31. $1, 2, 4$. **32.** $1, -2, -2$. **33.** $1, -i, i$.
34. $2, -2, -2$. **35.** $-1, 0, 1, 2$. **36.** $-2, -1, 1, 2$.

116. Limits for the Roots. In solving for the real roots of an equation, as we shall do, it is helpful to know a range of values which contains all the roots. Any number which is larger than or equal to the greatest root of an equation is called an **upper limit** for the roots. Any number which is smaller than or equal to the least root is called a **lower limit** for the roots.

Upper and lower limits can be determined by applying the following theorem.

Theorem. *If in the synthetic division of a polynomial $f(x)$ by $x - r$, where r is positive, each term of the third line is positive (some may be zero), then r is an upper limit for the real roots of the equation $f(x) = 0$. If r is negative and the terms of the third line are alternately positive and negative, then r is a lower limit for the roots.*

The truth of the theorem may be observed immediately. If a positive r produces positive terms in the third line in the division process, a number larger than r would increase all terms of the line after the first. The last term of the line could not then be zero. Consequently a value larger than r is not a root of the equation.

By quite similar reasoning the second part of the theorem can be established.

Example. Find upper and lower limits for the roots of the equation

$$2x^3 - 5x^2 - 7x + 4 = 0.$$

Solution. We show the tests for $r = 3$ and $r = 4$.

$$\begin{array}{rrrr|l} 2 & -5 & -7 & +4 & 3 \\ & +6 & +3 & -12 & \\ \hline 2 & +1 & -4 & -8 & \end{array} \qquad \begin{array}{rrrr|l} 2 & -5 & -7 & +4 & 4 \\ & +8 & +12 & +20 & \\ \hline 2 & +3 & +5 & +24 & \end{array}$$

These tests show that the smallest integral upper limit, as determined by the theorem, is 4.

We next try negative integers, starting with -1.

$$\begin{array}{rrrr|l} 2 & -5 & -7 & +4 & -1 \\ & -2 & +7 & +0 & \\ \hline 2 & -7 & +0 & +4 & \end{array} \qquad \begin{array}{rrrr|l} 2 & -5 & -7 & +4 & -2 \\ & -4 & +18 & -22 & \\ \hline 2 & -9 & +11 & -18 & \end{array}$$

Since -2 makes the terms of the third line alternate in signs, this number is a lower limit. Hence all the real roots of the given equation lie between -2 and 4.

117. Descartes' Rule of Signs. A polynomial $f(x)$ with the terms written in the order of descending powers of x is said to have a **variation of sign** if two consecutive terms have opposite signs. Any missing power of x is to be disregarded (a zero coefficient is not supplied) in this definition.

Illustration: The polynomial $2x^5 - x^4 - 3x^3 + x^2 - 2$ has three variations of sign. There is one variation from $2x^5$ to $-x^4$, another from $-3x^3$ to $+x^2$, and a third from $+x^2$ to -2.

The variations of sign furnish some information concerning the roots of a polynomial equation. In this connection we state a theorem which is sometimes useful. Because of the difficulty, the proof is not given.

THEOREM (Descartes' rule of signs). *The number of positive roots of a polynomial equation* $f(x) = 0$ *is equal to the number of variations of sign in* $f(x)$ *or is less than that number by an even integer. The number of negative roots of* $f(x) = 0$ *is equal to the number of variations of sign in* $f(-x)$ *or is less than that number by an even integer.*

Although the proof of this rule is omitted, we shall apply it to some equations. According to the rule, one variation of sign means there is exactly one positive root. For variations in excess of one the rule does not furnish definite information. Four variations, for example, reveal that there are four, two, or no positive roots.

Example 1. Apply Descartes' rule of signs to the equation

$$x^3 + 7x^2 + 8x - 17 = 0.$$

Solution. The left member $f(x)$ has one variation of sign. Hence the equation has exactly one positive root. To test for negative roots, we replace x by $-x$ and have

$$f(-x) = -x^3 + 7x^2 - 8x - 17.$$

Here there are two variations of sign. According to the rule, the given equation has either two negative roots or no negative root. We conclude, therefore, that the original equation has one positive root and that the two remaining roots are either both negative or both imaginary. That is, the possibilities are:

a. One positive root and two negative roots.
b. One positive root and two imaginary roots.

Example 2. Determine the information which Descartes' rule provides for the equation

$$x^4 + 3x^3 + 4x^2 + 3x + 5 = 0.$$

Solution. There is no variation of sign in the left member, and consequently the equation has no positive root. We have $f(-x) = x^4 - 3x^3 + 4x^2 - 3x + 5$, which contains four variations of sign. Hence, by the rule, the following possibilities exist:

a. All roots negative.
b. Two roots negative and two imaginary.
c. All roots imaginary.

Exercise 78

For each equation **1–18** find the greatest integral lower limit and the least integral upper limit for the roots which the theorem of Section 116 will yield.

1. $2x^3 + 5x^2 - x - 2 = 0.$
2. $3x^3 - 5x^2 - 3x + 4 = 0.$
3. $x^3 - x^2 - 16x + 15 = 0.$
4. $4x^3 - x^2 - x + 8 = 0.$
5. $2x^3 - 8x^2 + 3x + 2 = 0.$
6. $4x^3 - 5x^2 - 6x + 4 = 0.$
7. $x^3 - x^2 - 11x - 4 = 0.$
8. $2x^3 - x^2 - 14x - 3 = 0.$
9. $2x^3 + 3x^2 + x + 7 = 0.$
10. $5x^3 - x^2 - 25x + 5 = 0.$
11. $2x^3 + x^2 - 50x - 24 = 0.$
12. $3x^3 + x^2 - 30x + 1 = 0.$
13. $x^4 - 3x^2 - 4 = 0.$
14. $x^4 - 7x^2 + 6x + 6 = 0.$
15. $4x^4 + 3x^3 - 2x^2 - x = 0.$
16. $2x^4 - x^2 - 7x + 10 = 0.$
17. $2x^4 - x^3 - 6x^2 - x - 8 = 0.$
18. $3x^4 + 3x^3 - 8x^2 + x - 3 = 0.$

Apply Descartes' rule of signs to each of the following equations. Where the rule does not yield definite information, state the various possibilities as to the types of roots.

19. $2x^3 + 5x - 1 = 0.$
20. $x^3 + 3x + 3 = 0.$
21. $4x^3 - 5x^2 - 3 = 0.$
22. $3x^3 + 3x^2 + 4 = 0.$
23. $x^3 + 2x^2 - 3x + 5 = 0.$
24. $x^3 + x^2 + 3x - 2 = 0.$
25. $3x^3 - x^2 + x - 4 = 0.$
26. $5x^3 - x^2 + x - 1 = 0.$
27. $x^4 + x^2 - 6 = 0.$
28. $x^4 + x^3 - 1 = 0.$
29. $x^4 + x^3 + x^2 + x - 7 = 0.$
30. $x^4 - x^3 + 2x^2 - x + 3 = 0.$
31. $x^4 + 2x^3 + x^2 + x + 4 = 0.$
32. $x^4 + x^3 + 4x^2 - x + 8 = 0.$
33. $x^5 - 3x^3 - 5x - 4 = 0.$
34. $x^6 - 2x^5 + 4x^3 - 5 = 0.$

118. Rational Roots. A rational number which is a root of an equation is called a **rational root.** We shall consider a quite simple method of finding the rational roots, if any, of an equation with integral coefficients.

THEOREM. *If a rational number b/c in its lowest terms is a root of the equation*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

where the coefficients are all integers, then b is a factor of a_n and c is a factor of a_0.

Proof: Assuming that b/c is a root, we have

$$a_0\left(\frac{b}{c}\right)^n + a_1\left(\frac{b}{c}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{b}{c}\right) + a_n = 0. \qquad (1)$$

Multiplying this equation by c^n/b and transposing the last term gives

$$a_0b^{n-1} + a_1b^{n-2}c + a_2b^{n-3}c^2 + \cdots + a_{n-1}c^{n-1} = -\frac{a_nc^n}{b}. \qquad (2)$$

The left member of equation (2), being made up of terms whose factors are integers, has an integral value. Hence, the right member

$$-\frac{a_n c^n}{b}$$

likewise yields an integer. By hypothesis b and c have no common integral factor greater than 1; then it must follow that a_n is divisible by b.

Now returning to equation (1), we multiply by c^{n-1}, transpose the first term, and obtain

$$a_1 b^{n-1} + a_2 b^{n-2} c + a_3 b^{n-3} c^2 + \cdots + a_{n-1} b c^{n-2} + a_n c^{n-1} = -\frac{a_0 b^n}{c}.$$

The left member of this equation, having an integral value, means that the fraction in the right member is expressible as an integer. Since b and c have no common integral factor greater than 1, a_0 is divisible by c.

For the case in which $a_0 = 1$, the rational root b/c is an integer. This is true because c, being a factor of a_0, would be equal to 1 or -1. Hence we have the following corollary.

Corollary. *Any rational root of the equation*

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0,$$

where the coefficients are integers, is an integer and a factor of a_n.

Example 1. Find all the rational roots of

$$f(x) = x^4 + x^3 + 4x^2 + 6x - 12 = 0.$$

Solution. The rational roots, if any, must be factors of -12; and the possibilities are

$$\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 4, \quad \pm 6, \quad \pm 12.$$

Testing 1 by synthetic division, we find $f(1) = 0$.

$$\begin{array}{rrrrr|l} 1 + & 1 + & 4 + & 6 - & 12 & \ 1 \\ & 1 + & 2 + & 6 + & 12 & \\ \hline 1 + & 2 + & 6 + & 12 + & 0 & \end{array}$$

Then $f(x) = (x - 1)(x^3 + 2x^2 + 6x + 12)$. Additional roots of $f(x) = 0$ may be more readily found from the equation

$$Q_1(x) = x^3 + 2x^2 + 6x + 12 = 0.$$

$Q_1(x) = 0$ is called the **first depressed equation.** This equation has no positive roots; for a positive value for x makes all the terms on the left positive and their sum, of course, not equal to zero. Trying negative values, and starting with -1, we find $Q_1(-1) = 7$.

$$\begin{array}{rrrr|l} 1 & +2 & +6 & +12 & \underline{-1} \\ & -1 & -1 & -5 & \\ \hline 1 & +1 & +5 & +7 & \end{array}$$

Passing to -2, we get $Q_1(-2) = 0$.

$$\begin{array}{rrrr|l} 1 & +2 & +6 & +12 & \underline{-2} \\ & -2 & +0 & -12 & \\ \hline 1 & +0 & +6 & +0 & \end{array}$$

Therefore,

$$Q_1(x) = (x + 2)(x^2 + 6).$$

The equation $Q_2(x) = x^2 + 6 = 0$ is called the **second depressed equation.** Its roots are $\pm i\sqrt{6}$, which are not real. Hence -2 and 1 are the rational roots of $f(x) = 0$.

Example 2. Find the rational roots of

$$f(x) = 2x^4 - 3x^3 + 2x^2 - x - 3 = 0.$$

Solution. Any rational root must have its numerator a factor of -3 and its denominator a factor of 2. The numbers

$$\pm\tfrac{1}{2}, \quad \pm 1, \quad \pm\tfrac{3}{2}, \quad \pm 3$$

give all possible values for rational roots. Testing $\frac{3}{2}$, we find $f(\frac{3}{2}) = 0$.

$$\begin{array}{rrrrr|l} 2 & -3 & +2 & -1 & -3 & \underline{\tfrac{3}{2}} \\ & 3 & +0 & +3 & +3 & \\ \hline 2 & +0 & +2 & +2 & +0 & \end{array}$$

We next examine the depressed equation,

$$2x^3 + 2x + 2 = 0, \quad \text{or} \quad x^3 + x + 1 = 0,$$

for rational roots. A rational root of this equation must be a factor of 1. But neither 1 nor -1 satisfies the equation. Hence $\frac{3}{2}$ is the only rational root of the given equation.

Exercise 79

Find all the rational roots, and solve completely if the last depressed equation is quadratic.

1. $x^3 - x^2 + 4x - 4 = 0.$
2. $x^3 + x^2 + 3x + 3 = 0.$
3. $x^3 - 7x + 6 = 0.$
4. $x^3 + x^2 - 8x - 12 = 0.$
5. $x^3 + 2x^2 - 7x + 4 = 0.$
6. $x^3 - 2x - 4 = 0.$
7. $x^3 - 5x^2 + 8x - 6 = 0.$
8. $x^3 - 6x^2 + 21x - 26 = 0.$
9. $2x^3 - x^2 + 8x - 4 = 0.$
10. $3x^3 - x^2 + 3x - 1 = 0.$
11. $3x^3 - x^2 + 27x - 9 = 0.$
12. $4x^3 + x^2 + 16x + 4 = 0.$
13. $8x^3 + 4x^2 - 2x - 1 = 0.$
14. $8x^3 - 4x^2 - 2x + 1 = 0.$
15. $2x^3 - 7x^2 + 16x - 15 = 0.$
16. $3x^3 + 8x^2 + 19x + 10 = 0.$
17. $x^4 - 2x^3 + x^2 - x - 2 = 0.$
18. $x^4 + 2x^3 + x^2 + 3x + 2 = 0.$
19. $x^4 + 5x^3 + 7x^2 + 5x + 6 = 0.$
20. $x^4 - 8x^2 - 5x + 6 = 0.$
21. $2x^4 - 13x^3 + 30x^2 - 28x + 8 = 0.$
22. $x^4 - 5x^3 + 2x^2 + 8x = 0.$
23. $3x^4 + x^3 + 3x^2 + x = 0.$
24. $8x^4 - 24x^3 + 22x^2 - 12x + 9 = 0.$
25. $6x^4 + x^3 + 5x^2 + x - 1 = 0.$
26. $6x^4 + 5x^3 + 5x - 6 = 0.$

Factor the following polynomials:

27. $6x^3 - 5x^2 - 7x + 4.$
28. $4x^3 - 4x^2 - 3x - 10.$
29. $2x^3 - 7x^2 + 18x + 11.$
30. $5x^4 - 2x^3 - 2x.$
31. $x^4 + 4x^3 + 3x^2 - 4x - 4.$
32. $x^4 - 4x^3 + x^2 + 16x - 20.$
33. $x^5 + x^4 - 5x^3 - x^2 + 8x - 4.$
34. $x^5 + x^4 - 5x^3 - 5x^2 + 4x + 4.$

119. A Method of Approximating Irrational Roots. There are several processes by which the irrational roots of a polynomial equation may be found to any desired degree of accuracy. We shall consider a method which is called the solution by **successive enlargements.** This method employs the principle that if $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has a root between a and b. Graphically, this means that the curve $y = f(x)$ crosses the x-axis between any two of its points which are on opposite sides of the x-axis. The process consists in finding pairs of numbers closer and closer together for which $f(x)$ has opposite signs. We shall illustrate with some examples.

Example 1. Find to two decimal places the real root of the equation

$$f(x) = x^3 - 2x^2 + x - 1 = 0.$$

Solution. The graph (Fig. 22) of $y = f(x)$ shows that the given equation has a root between $x = 1$ and $x = 2$. The next step is to find consecutive tenths of units between which the curve crosses the x-axis. Computing, we find $f(1.6) = -0.424$. Next try 1.7 and 1.8:

$$\begin{array}{r|l}
1 - 2.0 + 1.00 - 1.000 & 1.7 \\
1.7 - 0.51 + 0.833 & \\
\hline
1 - 0.3 + 0.49 - 0.167 &
\end{array}$$

$$\begin{array}{r|l}
1 - 2.0 + 1.00 - 1.000 & 1.8 \\
1.8 - 0.36 + 1.152 & \\
\hline
1 - 0.2 + 0.64 + 0.152 &
\end{array}$$

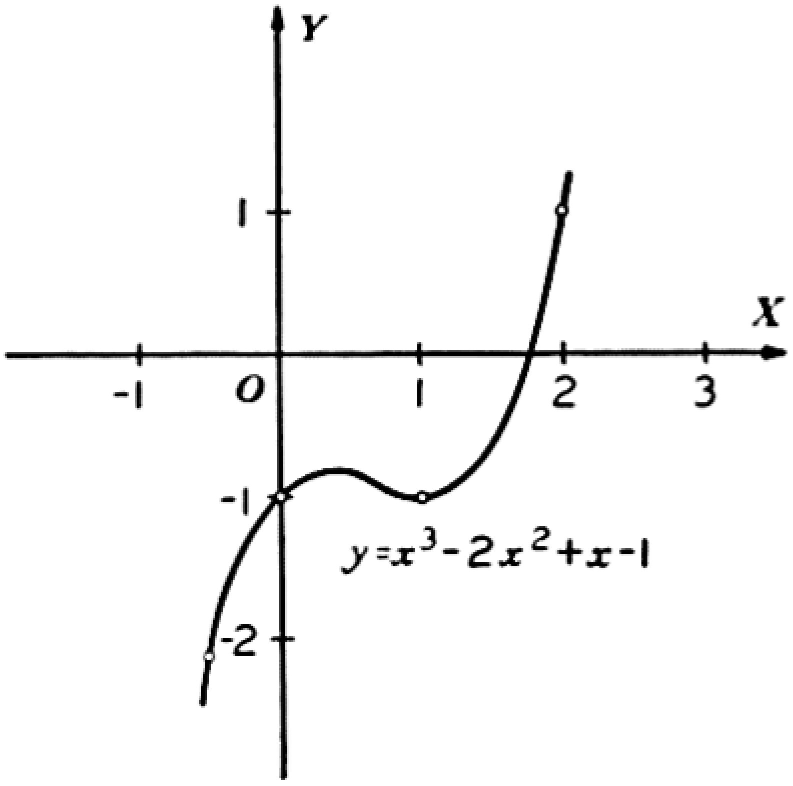


FIG. 22

We see that $f(1.7) = -0.167$ and $f(1.8) = 0.152$. Hence, the points $(1.7, -0.167)$ and $(1.8, 0.152)$ are on the graph of $y = f(x)$, one below the x-axis and the other above the x-axis. Consequently the desired root is between 1.7 and 1.8.

We next locate the root between consecutive hundredths of units. For this purpose we plot the points of the graph for which $x = 1.7$ and $x = 1.8$

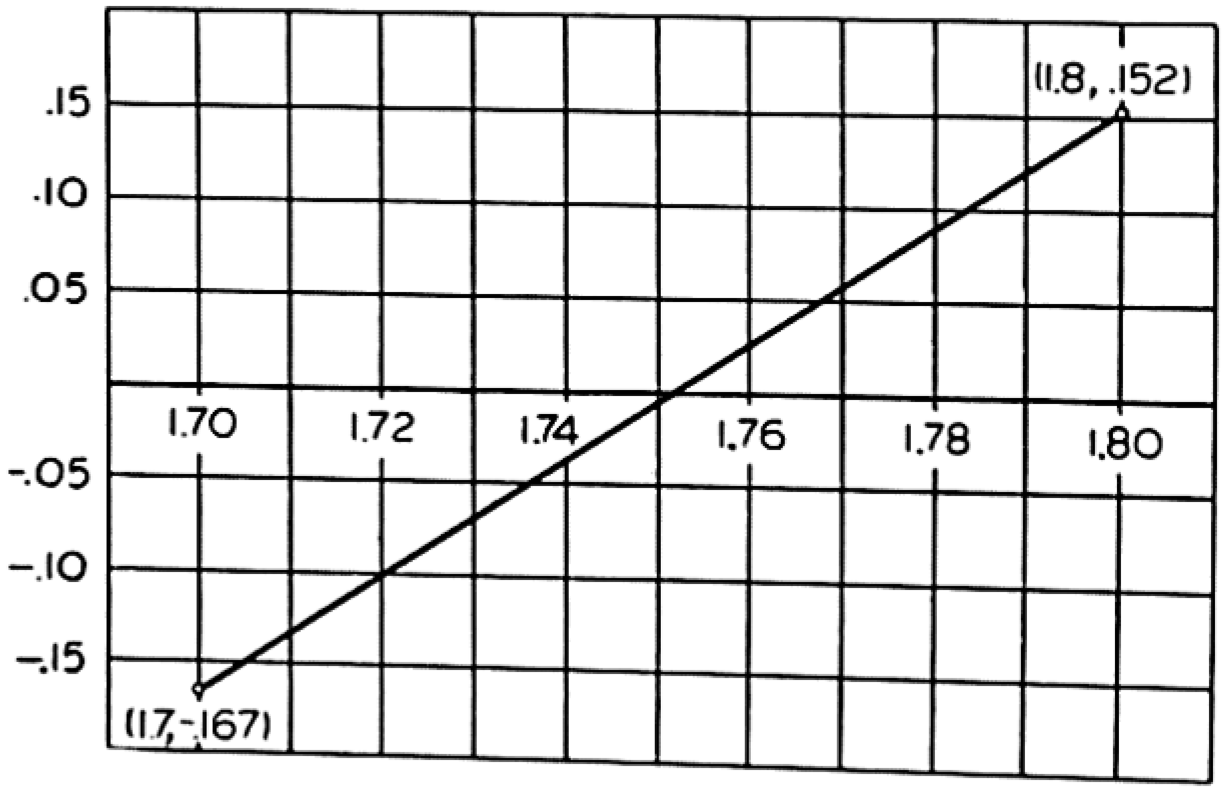


FIG. 23

on an enlarged scale (Fig. 23). The line joining these points crosses the x-axis at about $x = 1.75$, and the graph must cross the axis at a point quite near. By synthetic division we find $f(1.75) = -0.01526$ and $f(1.76) = 0.016576$.

$$\begin{array}{rrrr|l} 1 & -\ 2.00 & +\ 1.0000 & -\ 1.000000 & 1.75 \\ & 1.75 & -\ 0.4375 & +\ 0.984375 & \\ \hline 1 & -\ 0.25 & +\ 0.5625 & -\ 0.015625 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & -\ 2.000 & +\ 1.0000 & -\ 1.000000 & 1.76 \\ & 1.76 & -\ 0.4224 & +\ 1.016576 & \\ \hline 1 & -\ 0.24 & +\ 0.5776 & +\ 0.016576 & \end{array}$$

Thus we see that the root is between 1.75 and 1.76. Hence the root to three digits is 1.75. This process could be continued to obtain additional digits. The computations, of course, would become increasingly tedious.

It is easy, however, to estimate one additional digit graphically. Figure 24 shows the points on the graph for which $x = 1.75$ and $x = 1.76$. The line connecting these points crosses the x-axis at x about 1.755. Hence this value is approximately equal to a root of the given equation.

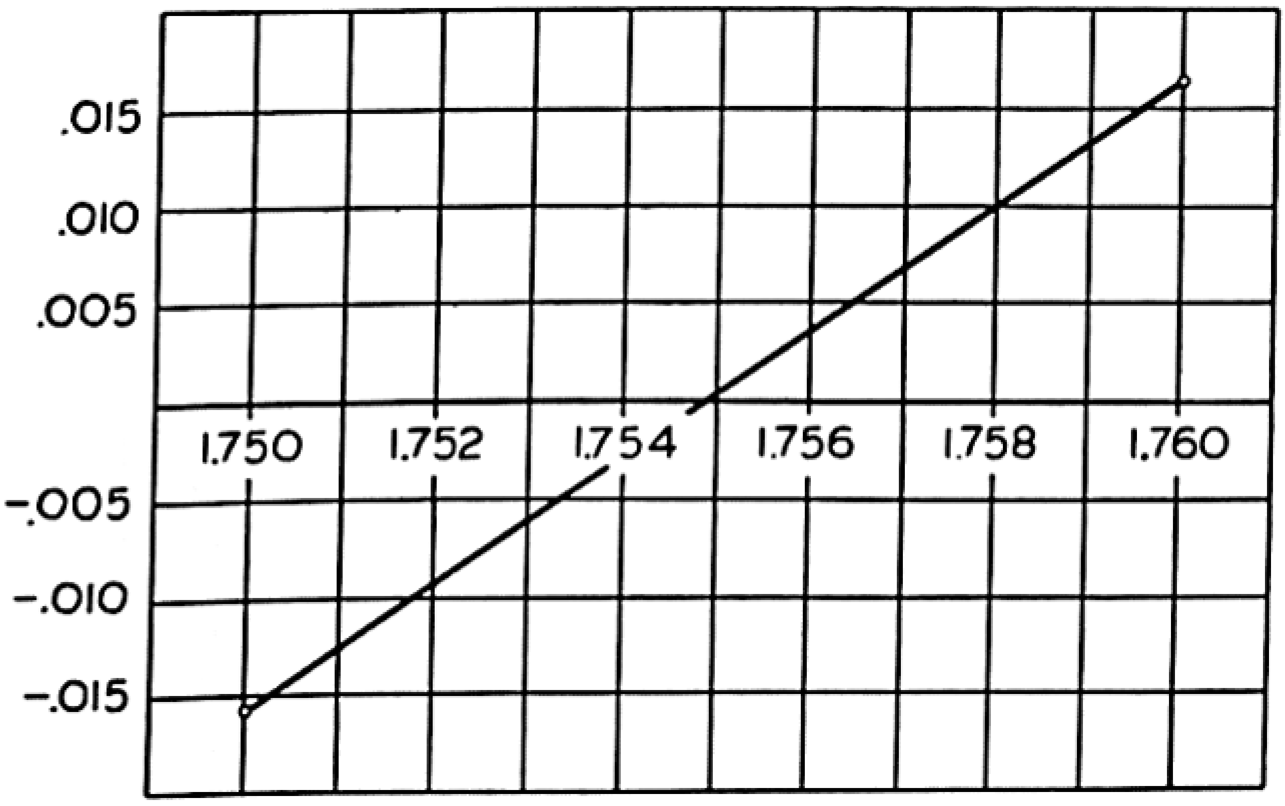


FIG. 24

Example 2. Find to two decimal places each of the real roots of

$$f(x) = x^3 + 3x^2 - x - 4 = 0.$$

Solution. A rough graph of $f(x)$ may be made by using a few integral values of x. The graph crosses the x-axis between -3 and -2; -2 and -1; and again between 1 and 2. The desired accuracy in the approxima-

x	$f(x)$
-3	-1
-2	2
-1	-1
0	-4
1	-1
2	14

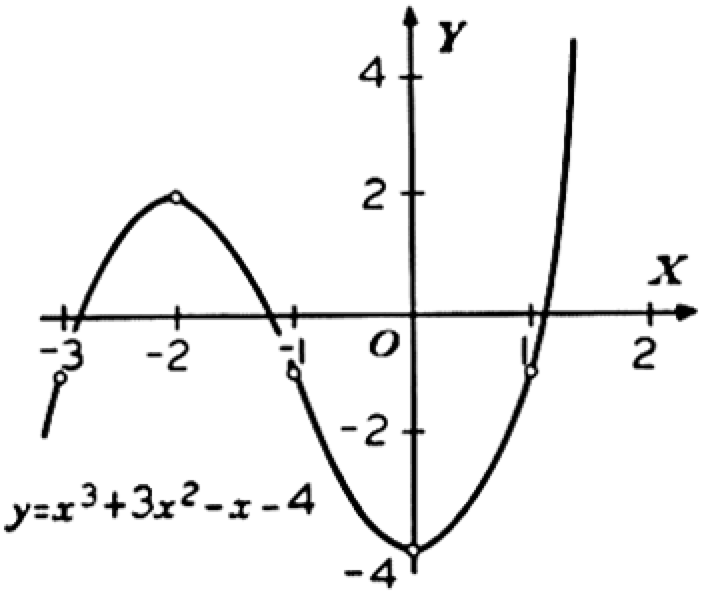


FIG. 25

tions to these roots may be obtained by considering each separately. We show the necessary computations for the smallest of the three roots.

$$\begin{array}{rrrr|l} 1 & +\,3.0 & -\,1.00 & -\,4.000 & \underline{-2.9} \\ & -\,2.9 & -\,0.29 & +\,3.741 & \\ \hline 1 & +\,0.1 & -\,1.29 & -\,0.259 & \end{array}$$

$$\begin{array}{rrrr|l} 1 & +\,3.0 & -\,1.00 & -\,4.000 & \underline{-2.8} \\ & -\,2.8 & -\,0.56 & +\,4.368 & \\ \hline 1 & +\,0.2 & -\,1.56 & +\,0.368 & \end{array}$$

The root lies between -2.9 and -2.8. From Figure 26 we estimate the third digit, and have $x = -2.86$. The other roots to two decimal places, as the reader should verify, are $x = -1.25$ and $x = 1.12$.

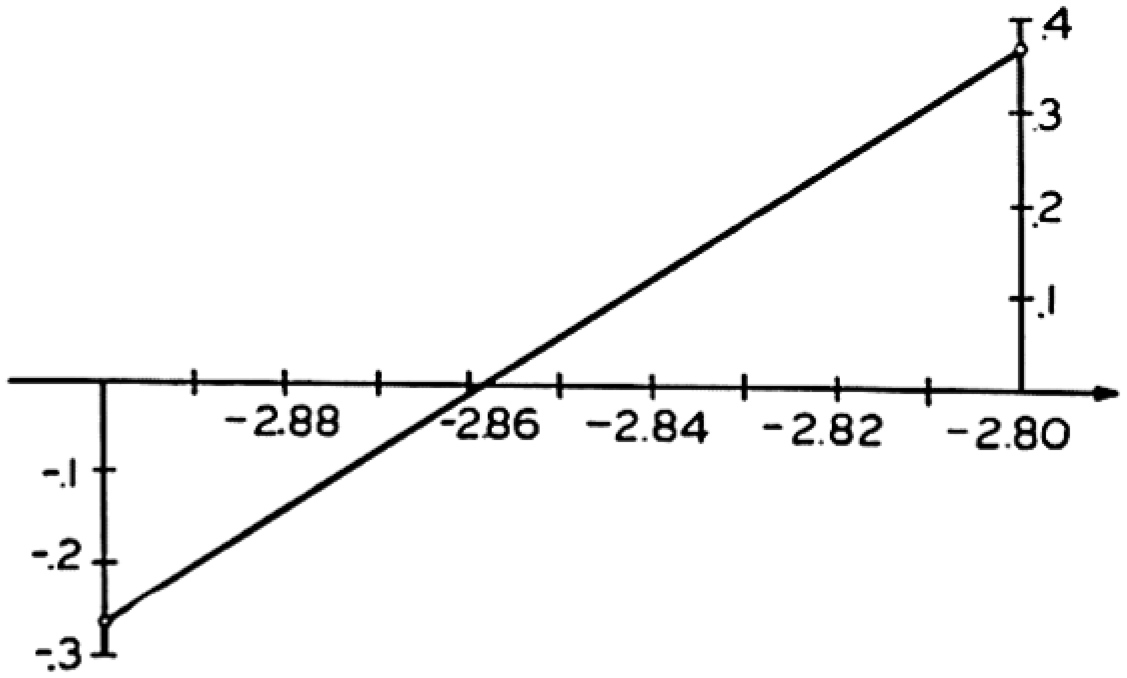


FIG. 26

In the two examples we have found approximations to the real roots of cubic equations. The process, of course, is applicable to equations of higher degrees. Further, a root can be obtained to as many decimal places as desired. It is evident, however, that the computations increase in tediousness with the degree of the equation and with the accuracy required in the result.

Exercise 80

Find to two decimal places the real root of each of the following equations:

1. $x^3 + 3x - 8 = 0$.
2. $x^3 + x - 14 = 0$.
3. $x^3 - x^2 - 2 = 0$.
4. $x^3 - 3x^2 + 4x - 3 = 0$.
5. $x^3 + 3x^2 + 3x - 27 = 0$.
6. $x^3 + 2x - 5 = 0$.
7. $x^3 + 3x^2 + 5x - 2 = 0$.
8. $x^3 - 2x^2 - 8 = 0$.
9. $x^3 + 3x^2 + 3x - 8 = 0$.
10. $2x^3 + 6x^2 + 6x - 3 = 0$.
11. $3x^3 + 4x^2 - x - 14 = 0$.
12. $2x^3 - 3x^2 - 3x - 6 = 0$.

Find three roots to two decimal places:

13. $x^3 - 3x^2 - x + 4 = 0$.
14. $x^3 - 3x^2 - 2x + 1 = 0$.
15. $x^3 - 6x^2 + 8x + 1 = 0$.
16. $x^3 - 5x^2 + 3x + 8 = 0$.
17. $x^3 - 4x^2 + 3x + 1 = 0$.
18. $x^3 - 9x^2 + 24x - 17 = 0$.
19. $4x^3 - 12x^2 + 4x + 5 = 0$.
20. $3x^3 - 5x^2 - 4x + 3 = 0$.

Find the indicated root to two decimal places:

21. $x^4 - x - 3 = 0$; the root between 1 and 2.
22. $x^4 + 2x^3 - 2 = 0$; the root between 0 and 1.
23. $x^4 + x^3 - 6x - 15 = 0$; the root between 2 and 3.
24. $x^4 - x^3 + x^2 - 3x - 4 = 0$; the root between 1 and 2.

120. Transformations of Equations. If the equation $f(x) = 0$ has the unknown x replaced by another expression, a new equation is obtained. That is, the given equation is **transformed** into another equation. We shall consider two particular ways of transforming equations. These transformations will be applied to polynomial equations, or rational integral equations, and will be used in approximating irrational roots by the method discussed in the following section.

A. *Transformation to change the sign of each root.* If we substitute $x = -y$ in the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

we obtain

$$a_0(-y)^n + a_1(-y)^{n-1} + \cdots + a_{n-1}(-y) + a_n = 0.$$

It may be readily noticed that if r is a root of the first equation, then $-r$ is a root of the transformed equation. Also it may be noticed that the equation in y may be written so that it and the given equation have equal coefficients for all corresponding even powers of the unknowns, and that the corresponding odd powers of the unknowns have coefficients numerically equal but opposite in sign. Hence we state the following rule.

RULE. *To obtain an equation each root of which is numerically equal to a corresponding root of a given equation, but opposite in sign, change the signs of the coefficients of the odd powers of the unknown.*

Example 1. Form an equation whose roots are numerically equal but opposite in sign to the roots of

$$x^4 - 2x^3 - 13x^2 + 38x - 24 = 0.$$

Solution. Following the rule, we obtain at once

$$x^4 + 2x^3 - 13x^2 - 38x - 24 = 0.$$

It may be verified that 1, 2, 3, -4 are the roots of the given equation, and that those of the transformed equation are -1, -2, -3, 4.

B. *Transformation to decrease each root of a given equation by a constant.* By substituting $x = y + h$ in the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0, \tag{1}$$

we get

$$a_0(y + h)^n + a_1(y + h)^{n-1} + \cdots + a_{n-2}(y + h)^2 + a_{n-1}(y + h) + a_n = 0. \tag{2}$$

By examining these equations it may be observed that if r is a root of the given equation, then the transformed equation is satisfied by $y = r - h$. Consequently, each root of the given equation decreased by h is a root of the new equation.

If the binomials of equation (2) are expanded and like powers of y are collected, the resulting equation is of the form

$$a_0y^n + A_1y^{n-1} + \cdots + A_{n-2}y^2 + A_{n-1}y + A_n = 0. \tag{3}$$

To find the unknown coefficients $A_1, A_2, \cdots, A_n$, we use the device of substituting $y = x - h$. That is, we write

$$a_0(x - h)^n + A_1(x - h)^{n-1} + \cdots + A_{n-2}(x - h)^2 + A_{n-1}(x - h) + A_n = 0. \tag{4}$$

Now this second substitution gives an equation whose left member is identically equal to the left member of the given equation (1), and could be written in the same form. Then if we divide each of these equations by $x - h$, the remainder in each case is the same. From equation (4) it may be seen that the remainder is A_n. The quotient is

$$a_0(x-h)^{n-1} + A_1(x-h)^{n-2} + \cdots + A_{n-2}(x-h) + A_{n-1}.$$

Dividing this quotient by $x - h$, we obtain the second remainder A_{n-1}; and by dividing the second quotient by $x - h$, we have the remainder A_{n-2}. This process may be continued until A_1 is obtained as a remainder. Then if we use the given equation and carry through the divisions here described, the coefficients $A_1, A_2, \cdots, A_n$ may be had from the set of remainders. Hence we state the following rule.

Rule. *To transform a given equation $f(x) = 0$ into an equation each root of which is a constant h less than the corresponding root of the given equation, take the following steps:*

1. *Divide $f(x)$ by $x - h$.*
2. *Setting the remainder aside, divide the quotient of step 1 by $x - h$.*
3. *Divide the quotient of step 2 by $x - h$, and continue the process until a constant quotient (a_0) is obtained.*
4. *Start with the final quotient and then take the remainders in the reverse order from which they were obtained and write these numbers as the coefficients of the powers of the unknown in descending order.*

The roots of an equation may be increased by decreasing them by a negative number. That is, to increase the roots by h the divisions would be made by using $x + h$ as the divisor.

Example 2. Obtain an equation each root of which is 3 less than the corresponding root of the equation

$$f(x) = 2x^3 - 4x^2 + 7x - 6 = 0.$$

Solution. We divide $f(x)$ and the successive quotients by $x - 3$ to compute the coefficients of the transformed equation.

$$\begin{array}{lrrrll}
 & 2\ - & 4\ + & 7\ - & 6 & \underline{|\ 3} \\
 & + & 6\ + & 6\ + & 39 & \\ \cline{2-5}
\text{First quotient:} & 2\ + & 2\ + & \multicolumn{1}{r|}{13} & +33 & \qquad A_3 = 33. \\
 & + & 6\ + & \multicolumn{1}{r|}{24} & & \\ \cline{2-4}
\text{Second quotient:} & 2\ + & \multicolumn{1}{r|}{8} & +\ 37 & & \qquad A_2 = 37. \\
 & + & \multicolumn{1}{r|}{6} & & & \\ \cline{2-3}
\text{Constant quotient:} & 2\ + & 14 & & & \qquad A_1 = 14.
\end{array}$$

The required equation is $2x^3 + 14x^2 + 37x + 33 = 0$. The roots of this equation are equal to the roots of the given equation decreased by 3.

Exercise 81

Write equations whose roots are numerically equal but opposite in sign to the roots of the following equations:

1. $x^2 - 3x + 5 = 0.$
2. $3x^2 + 6x - 1 = 0.$
3. $2x^3 + 4x^2 + x + 3 = 0.$
4. $2x^3 - 3x + 4 = 0.$
5. $x^4 + 6x^3 - 4x^2 - 2x - 1 = 0.$
6. $x^4 - 3x^3 - 2x^2 - x - 3 = 0.$

Form equations whose roots are equal to the roots of the following equations less the number indicated:

7. $x^2 - 9x + 20 = 0; 3.$
8. $x^2 - 9x + 20 = 2; 2.$
9. $x^3 + 3x^2 + 4x - 12 = 0; 1.$
10. $x^3 - 5x^2 + 7x - 8 = 0; 2.$
11. $3x^3 - 5x^2 - 12 = 0; 1.$
12. $x^3 - 4x^2 - 6x - 2 = 0; 4.$
13. $4x^3 + 8x^2 + 5x + 3 = 0; -1.$
14. $2x^3 + 7x^2 - 9 = 0; -3.$
15. $2x^3 - 3x^2 - 10 = 0; 3.$
16. $x^3 - 5x^2 + 3x - 2 = 0; 0.2.$
17. $x^3 - 10x^2 - 1 = 0; 0.5.$
18. $3x^3 - 5x^2 - 4x + 3 = 0; 0.6.$

121. Horner's Method of Approximating Irrational Roots. In Section 119 irrational roots of polynomial equations were approximated by the method of successive enlargements. We shall now show another way of computing irrational roots to any desired degree of accuracy. This method, which we shall explain in an example, is called **Horner's method.**

Example. Find to three decimal places the positive root and the numerically larger negative root of the equation

$$f(x) = x^3 + 3x^2 - x - 4 = 0.$$

Solution. The graph of the function $y = x^3 + 3x^2 - x - 4$ is shown in Figure 25, Section 119. From the graph we notice that the positive root is between 1 and 2, and that the desired negative root is between -2 and -3. We shall solve first for the positive root, using the following procedure:

First, transform the given equation into an equation whose roots are diminished by 1.

$$
\begin{array}{rrrl}
1+3 & -1 & -4 & \multicolumn{1}{|l}{1} \\
+1 & +4 & +3 & \\ \cline{1-3}
1+4 & \multicolumn{1}{r|}{+3} & -1 & \\
+1 & \multicolumn{1}{r|}{+5} & & \\ \cline{1-2}
\multicolumn{1}{r|}{1+5} & +8 & & \\
\multicolumn{1}{r|}{+1} & & & \\ \cline{1-1}
1+6 & & &
\end{array}
$$

The transformed equation is

$$f_1(x_1) = x_1^3 + 6x_1^2 + 8x_1 - 1 = 0.$$

The roots of this equation are equal to the roots of $f(x) = 0$ diminished by 1. Hence the new equation has a root between 0 and 1 which corresponds to the root between 1 and 2 of the given equation. We next find the successive tenths between which the root of $f_1(x_1) = 0$ lies. To get a suggestion as to the approximate location of the root we neglect for the moment the first two terms of $f_1(x_1) = 0$. These two terms are likely small in comparison with the other terms since they contain higher powers of a number between 0 and 1. Then, omitting the first two terms, we write

$$8x_1 - 1 = 0 \quad \text{and} \quad x_1 = 0.1^+.$$

This rough approximation is called the **trial divisor.** Then trying the trial divisor and values near it, we find

$$f_1(0.1) = -0.139 \quad \text{and} \quad f_1(0.2) = 0.848.$$

Since $f_1(x_1) = 0$ has a root between 0.1 and 0.2, the next step is to transform the equation so that the roots are decreased by 0.1.

$$\begin{array}{rrrr|l}
1 & +\,6.0 & +\,8.00 & -\,1.000 & 0.1 \\
 & +\,0.1 & +\,0.61 & +\,0.861 & \\ \cline{1-4}
1 & +\,6.1 & +\,8.61 & -\,0.139 & \\
 & +\,0.1 & +\,0.62 & & \\ \cline{1-3}
1 & +\,6.2 & +\,9.23 & & \\
 & +\,0.1 & & & \\ \cline{1-2}
1 & +\,6.3 & & &
\end{array}$$

The second transformed equation is

$$f_2(x_2) = x_2^3 + 6.3x_2^2 + 9.23x_2 - 0.139 = 0.$$

This equation has a root between 0 and 0.1 which corresponds to the root of $f_1(x_1) = 0$ between 0.1 and 0.2. We now find a trial divisor from the equation $9.23x_2 - 0.139 = 0$ and get $x_2 = 0.01+$. Computing, we get

$f_2(0.01) = -0.046069$ and $f_2(0.02) = 0.048128$. The root being between 0.01 and 0.02, we transform $f_2(x_2) = 0$ to decrease the roots by 0.01.

$$
\begin{array}{rrrl}
1 + 6.30 & + 9.2300 & - 0.139000 & \multicolumn{1}{|l}{0.01} \\
0.01 & + 0.0631 & + 0.092931 & \multicolumn{1}{|l}{} \\ \cline{4-4}
\cline{1-3}
1 + 6.31 & \multicolumn{1}{r|}{+ 9.2931} & - 0.046069 & \\
+ 0.01 & \multicolumn{1}{r|}{+ 0.0632} & & \\
\cline{1-2}
\multicolumn{1}{r|}{1 + 6.32} & + 9.3563 & & \\
\multicolumn{1}{r|}{0.01} & & & \\
\cline{1-1}
1 + 6.33 & & &
\end{array}
$$

Thus we obtain the third transformed equation

$$f_3(x_3) = x_3^3 + 6.33x_3^2 + 9.3563x_3 - 0.046069 = 0.$$

This equation has a root between 0 and 0.01 corresponding to the root of $f_2(x_2) = 0$ between 0 and 0.1. Using the last two terms of $f_3(x_3) = 0$, we find $x_3 = 0.0049+$, or, rounding off, $x_3 = 0.005$.

From these successive transformations we know that the root between 1 and 2 of the given equation is approximately

$$x = 1 + 0.1 + 0.01 + 0.005 = 1.115.$$

The first three digits are correct, and the 5 appears to be the best digit for the third decimal place. If greater accuracy is desired, the process can be continued.

To compute the negative root between -2 and -3, we transform $f(x) = 0$ into an equation whose roots are of the same absolute value but of opposite signs. That is, we solve the equation

$$f(-x) = -x^3 + 3x^2 + x - 4 = 0 \quad \text{or} \quad x^3 - 3x^2 - x + 4 = 0,$$

for its root between 2 and 3 and take the negative of the value so obtained. Proceeding in just the same way as before, but omitting the discussion, we put the computations in the compact form shown below. It may be noticed that a trial divisor from the first transformed equation is not helpful. This is due to the fact that the root of the equation is not near enough to zero to make the higher degree terms small in comparison with the last two terms.

$$\begin{array}{rrrrl}
1 & -3 & -1 & +4 & \multicolumn{1}{|l}{2} \\
 & +2 & -2 & -6 & \multicolumn{1}{|l}{} \\ \cline{1-4} \cline{5-5}
1 & -1 & \multicolumn{1}{r|}{-3} & -2 & \\
 & +2 & \multicolumn{1}{r|}{+2} & & \\ \cline{1-3}
1 & \multicolumn{1}{r|}{+1} & -1 & & \\
 & \multicolumn{1}{r|}{+2} & & & \\ \cline{1-2}
1 & +3 & & &
\end{array}$$

$$\begin{array}{rrrrl}
1 & +3.0 & -1.00 & -2.000 & \multicolumn{1}{|l}{0.8} \\
 & +0.8 & +3.04 & +1.632 & \multicolumn{1}{|l}{} \\ \cline{1-4} \cline{5-5}
1 & +3.8 & \multicolumn{1}{r|}{+2.04} & -0.368 & \\
 & +0.8 & \multicolumn{1}{r|}{+3.68} & & \\ \cline{1-3}
1 & \multicolumn{1}{r|}{+4.6} & +5.72 & & \\
 & \multicolumn{1}{r|}{+0.8} & & & \\ \cline{1-2}
1 & +5.4 & & &
\end{array}$$

$$5.72x_2 - 0.368 = 0.$$
$$x_2 = 0.06+.$$

$$\begin{array}{rrrrl}
1 & +5.40 & +5.7200 & -0.368000 & \multicolumn{1}{|l}{0.06} \\
 & 0.06 & +0.3276 & +0.362856 & \multicolumn{1}{|l}{} \\ \cline{1-4} \cline{5-5}
1 & +5.46 & \multicolumn{1}{r|}{+6.0476} & -0.005144 & \\
 & +0.06 & \multicolumn{1}{r|}{+0.3312} & & \\ \cline{1-3}
1 & \multicolumn{1}{r|}{+5.52} & +6.3788 & & \\
 & \multicolumn{1}{r|}{+0.06} & & & \\ \cline{1-2}
1 & +5.58 & & &
\end{array}$$

$$f_3(x_3) = x_3{}^3 + 5.58x_3{}^2 + 6.3788x_3 - 0.005144 = 0.$$

$$6.3788x_3 - 0.005144 = 0, \quad \text{and} \quad x_3 = 0.0008+, \quad \text{or} \quad 0.001.$$

Hence $x = -(2 + 0.8 + 0.06 + 0.001) = -2.861$, approximately.

Having found the approximations 1.115 and -2.861, we can quickly obtain an approximation to the third root. The sum of the three roots of the given equation is -3 (Section 115). Hence, if r stands for the third root, we have $r + 1.115 - 2.861 = -3$, and $r = -1.254$.

Exercise 82

Use Horner's method to find a root to three decimal places:

1. $x^3 + 3x^2 + 6x - 4 = 0.$
2. $x^3 + 3x^2 + 4x - 12 = 0.$
3. $x^3 + x - 1 = 0.$
4. $x^3 - 5x^2 + 7x - 8 = 0.$
5. $x^3 - 3x^2 + 5x - 8 = 0.$
6. $x^3 - 5x^2 + 7x - 11 = 0.$
7. $3x^3 - 5x^2 - 12 = 0.$
8. $2x^3 + 3x^2 - 3x - 10 = 0.$
9. $4x^3 + 12x^2 + 12x - 5 = 0.$
10. $2x^3 - 3x^2 - 10 = 0.$

Use Horner's method to find all the roots to three decimal places; check by adding the roots.

11. $x^3 - 4x + 1 = 0.$
12. $x^3 - 5x - 3 = 0.$
13. $x^3 - 2x^2 - 4x + 7 = 0.$
14. $x^3 - x^2 - 2x + 1 = 0.$
15. $x^3 - x^2 - 4x + 2 = 0.$
16. $3x^3 + 4x^2 - 5x - 3 = 0.$
17. $4x^3 - 8x + 1 = 0.$
18. $2x^3 + 5x^2 + x - 1 = 0.$

Each of the following equations has two real roots and two imaginary roots. Use Horner's method and compute the real roots to two decimal places.

19. $x^4 - 10x - 10 = 0.$
20. $x^4 - 6x^2 - 12x + 5 = 0.$
21. $x^4 + 3x^3 + 7x^2 + 9x + 1 = 0.$
22. $x^4 + x^2 + 3x - 4 = 0.$
23. $x^4 + 2x^2 - 3x - 5 = 0.$
24. $x^4 - x - 10 = 0.$

Chapter Sixteen

COMPLEX NUMBERS

122. The Fundamental Operations with Complex Numbers. We have already introduced complex numbers with imaginary parts and have performed the four fundamental operations on quantities of this kind. In this chapter we shall consider complex numbers in more detail. Before proceeding with this material, however, the student is urged to read carefully Section 64 again.

As we have seen, a quantity of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$, is a complex number. The operations of addition, subtraction, multiplication, and division of complex numbers are governed by the following definitions:

Sum: $(a + bi)+(c + di) = (a + c) + (b + d)i.$
Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i.$
Product: $(a + bi)(c + di) = (ac - bd) + (bc + ad)i.$

$$\text{Quotient: } \frac{a + bi}{c + di} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

These definitions will be better appreciated if we actually perform the operations indicated in the left members of the equations. The operations should be carried out just as if all quantities were real, with the understanding that i^2 is to be replaced by -1. The right members of the equations will then be obtained. This fact is immediately evident for the operations of addition and subtraction. For obtaining the product we have

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (bc + ad)i.\end{aligned}$$

The quotient is obtained by multiplying the dividend and the divisor by the conjugate of the divisor. Thus,

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

123. Graphical Representation of Complex Numbers. The complex numbers are usually represented by the points on the plane of a rectangular coordinate system. A number of the form $a + bi$ is represented by the point whose coordinates are (a, b). Thus any complex number determines a point in the plane. Conversely, any point of the plane determines a complex number. The points on the axes represent special complex numbers. Points on the x-axis represent real numbers because the ordinates of these points are zero. Similarly, the points on the y-axis have abscissas equal to zero and therefore correspond to pure imaginary numbers. When a rectangular coordinate plane is used to represent complex numbers the x-axis is called the **real axis,** the y-axis the **pure imaginary axis,** and the plane is called the **complex plane.**

In Figure 27, the correspondence between some complex numbers and points is indicated.

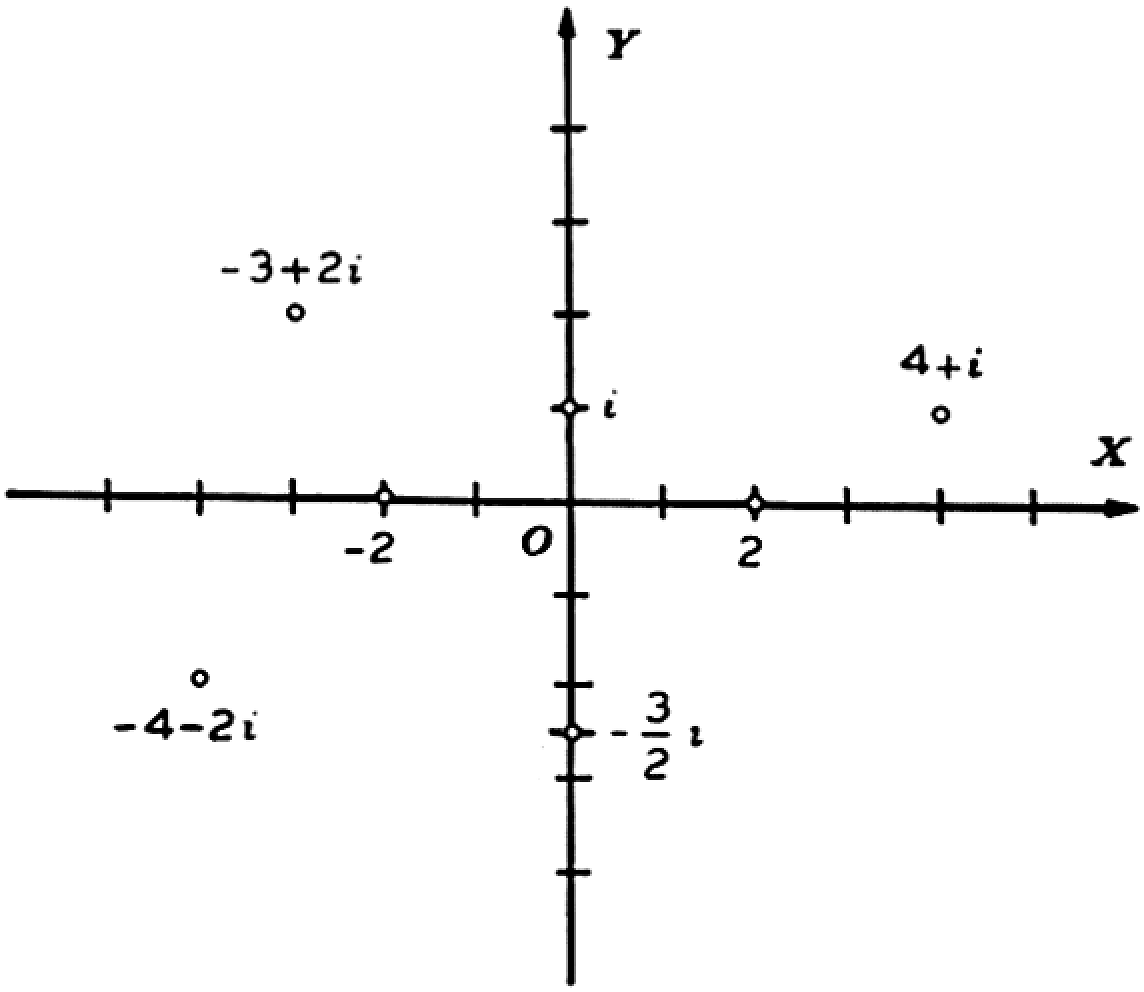


Fig. 27

The sum of two complex numbers may be found graphically by constructing a parallelogram. The two points representing the numbers are opposite vertices and the origin is the third vertex. The fourth vertex of the parallelogram thus determined represents the sum. This method of adding can be verified readily by an examination of Figure 28.

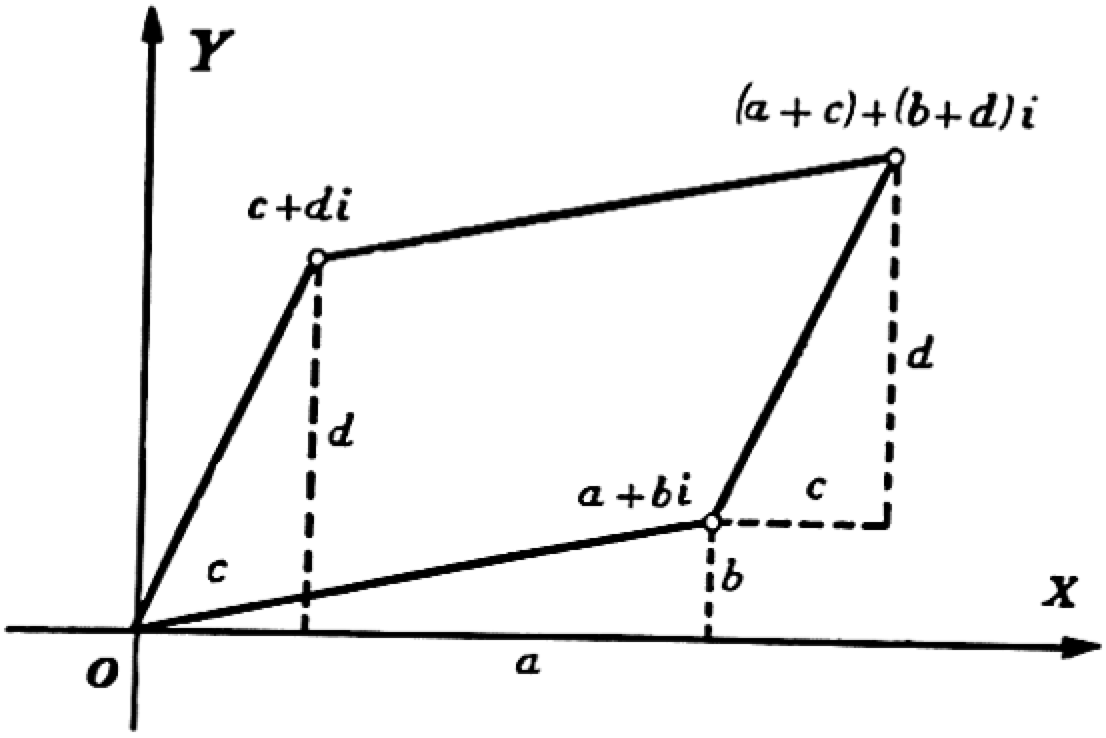


FIG. 28

The difference of two complex numbers is obtained graphically by changing the sign of the subtrahend and then adding.

124. The Polar Form of a Complex Number.* A complex number expressed as $a + bi$ is said to be in **rectangular form.** This name comes from the plan of representing the number in a rectangular coordinate plane. We next show how to express a complex number in another useful form.

The point P (Fig. 29) corresponding to the number $a + bi$ has the coordinates (a, b). The distance from the origin to P is denoted by r, and the angle from the positive real axis to the distance segment is denoted by θ. The distance r (chosen positive where P is any point other than the origin) is called the **absolute value,** or **modulus,** or the complex number. The angle θ is called the **amplitude,** or **argument.**

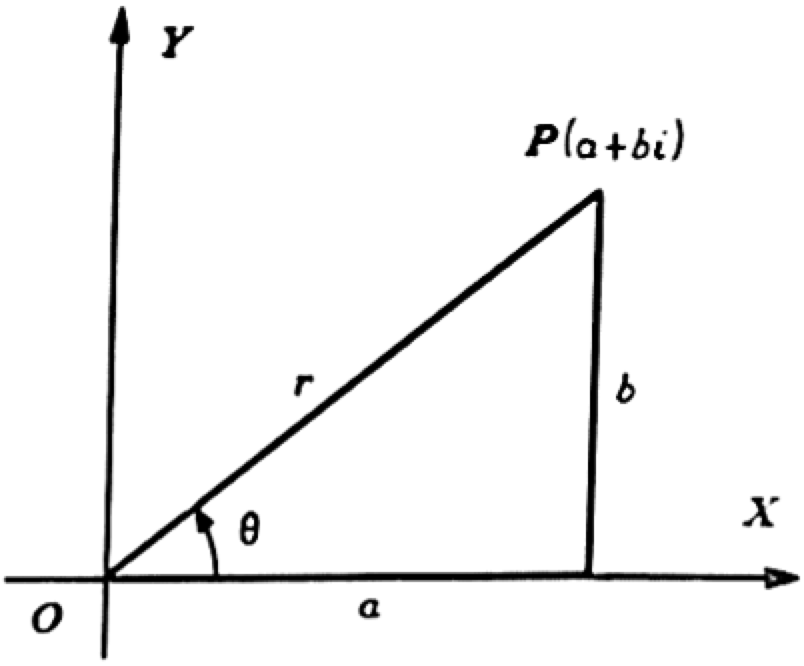


FIG. 29

Referring to the diagram, we have the relations

$$r = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \text{arc tan}\,\frac{b}{a}. \tag{1}$$

Further, we may write

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta. \tag{2}$$

Hence it follows that

$$a + bi = r\,(\cos\theta + i\sin\theta). \tag{3}$$

---

[*] An understanding of trigonometry is needed in the remainder of this chapter.

The expression $r(\cos\theta + i\sin\theta)$ is called the **polar,** or **trigonometric,** form of a complex number. The angle θ in this form may be changed by any integral multiple of 360° without altering the sine or cosine of the angle. Hence a complex number can be expressed in the more general polar form

$$r[\cos(\theta + k\cdot 360°) + i\sin(\theta + k\cdot 360°)],$$

where k is any integer. We shall soon see an important use of this form of a complex number.

We note that equations (1) and (2) may be employed to change a number from rectangular to polar form, and vice versa. In applying equations (1), however, the value of θ must be chosen to correspond to the quadrant of a given complex number. This is illustrated in the following example.

Example 1. Plot the number $2\sqrt{3} - 2i$ and change to polar form.

Solution. The point $(2\sqrt{3}, -2)$ represents the number. The absolute value and the amplitude (Fig. 30) are given by

$$r = \sqrt{12 + 4} = 4 \quad \text{and} \quad \theta = \text{arc tan}\,\frac{-2}{2\sqrt{3}} = 330°.$$

The tangent of 150° is also equal to $-2/2\sqrt{3}$. We reject this angle because its terminal side is in the second quadrant. The terminal side of 330° passes through the point which represents the given number. Hence

$$2\sqrt{3} - 2i = 4(\cos 330° + i\sin 330°).$$

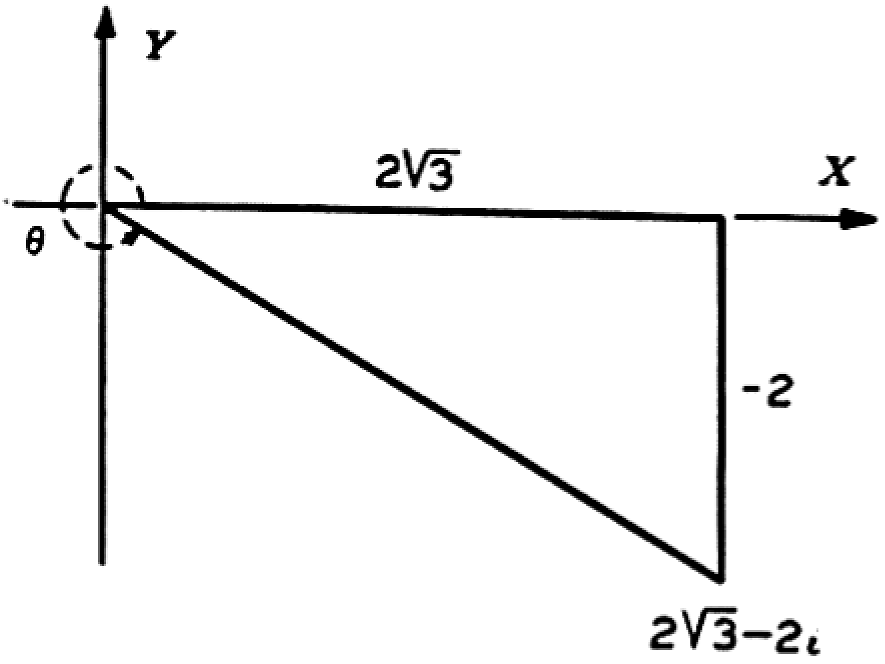


FIG. 30

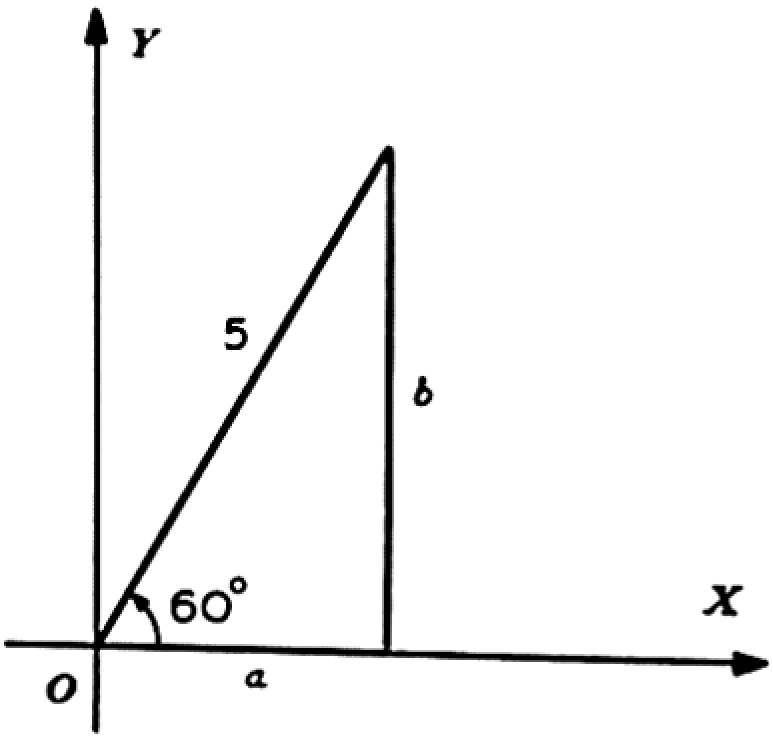


FIG. 31

Example 2. Plot the number $5(\cos 60° + i\sin 60°)$ and change to rectangular form.

Solution. The point corresponding to this number is easily located by drawing the angle 60° in standard position and measuring off 5 units along the terminal side. From equations (2), and also from Figure 31, we have

$$a = 5\cos 60° = 5(\tfrac{1}{2}) = \tfrac{5}{2}$$

and

$$b = 5\sin 60° = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}.$$

Hence

$$5(\cos 60° + i\sin 60°) = \frac{5}{2} + \frac{5\sqrt{3}}{2}i.$$

Example 3. Express the complex number $4 + 3i$ in polar form.
Solution. Since $a = 4$ and $b = 3$, we have

$$r = \sqrt{16 + 9} = 5 \quad \text{and} \quad \text{arc tan } \theta = 0.75.$$

In this case we need a table to find θ. Referring to Table IV, page 305, we find that θ, to the nearest degree, is 37°. Hence,

$$4 + 3i = 5\,(\cos 37° + i\sin 37°).$$

Exercise 83

Write the conjugate of each complex number. Plot the number and its conjugate:

1. $3 + 2i$.
2. $2 + 3i$.
3. $4 - 3i$.
4. $5 + 4i$.
5. $-2 + i$.
6. $6 - 5i$.
7. $-4 - 3i$.
8. $5i$.
9. $-i$.
10. 4.
11. -3.
12. $-4i$.

Perform the indicated operations graphically. Check each result by finding the sum or difference algebraically:

13. $(4 + i) + (1 + 3i)$.
14. $(-2 + 2i) + (3 + i)$.
15. $(3 - i) + (-2 - i)$.
16. $(7 - 3i) + (5 + 4i)$.
17. $(1 + i) - (3 - 5i)$.
18. $(6 + 3i) - (-2 + 3i)$.
19. $(8 + 6i) - (-1 + 4i)$.
20. $2 + (3 + 5i)$.
21. $(5 - 7i) - (3i)$.

Plot each number and write the corresponding polar form:

22. $1 + i$.
23. $4 - 4i$.
24. 4.
25. $1 + i\sqrt{3}$.
26. $2 - 2i\sqrt{3}$.
27. $-\sqrt{3} + i$.
28. $2i$.
29. -3.
30. $-i$.
31. $\sqrt{2} + i\sqrt{2}$.
32. $8\sqrt{3} - 8i$.
33. $-\sqrt{15} + i\sqrt{5}$.
34. $\sqrt{7} - i\sqrt{21}$.
35. $3 + 4i$.
36. $-2 + i$.
37. $5 - 2i$.
38. $-7 - 24i$.
39. $8 + 6i$.

Change each number to rectangular form:

40. $2(\cos 30^\circ + i \sin 30^\circ)$.
41. $8(\cos 60^\circ + i \sin 60^\circ)$.
42. $\sqrt{2}\,(\cos 45^\circ + i \sin 45^\circ)$.
43. $\cos 135^\circ + i \sin 135^\circ$.
44. $5(\cos 150^\circ + i \sin 150^\circ)$.
45. $7(\cos 180^\circ + i \sin 180^\circ)$.
46. $6(\cos 270^\circ + i \sin 270^\circ)$.
47. $20(\cos 90^\circ + i \sin 90^\circ)$.
48. $\cos 128^\circ + i \sin 128^\circ$.
49. $2(\cos 51^\circ + i \sin 51^\circ)$.

125. Multiplication and Division of Numbers in Polar Form. The product and quotient of two complex numbers may be written immediately if the numbers are in polar form. We let

$$r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad r_2(\cos\theta_2 + i\sin\theta_2)$$

stand for two complex numbers. Multiplying these numbers in the usual way and applying the trigonometric formulas for the sine and the cosine of the sum of two angles, we obtain

$$\begin{aligned}
&[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]\\
&\qquad = r_1r_2[\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2]\\
&\qquad = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]\\
&\qquad = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].
\end{aligned}$$

This formula for the product of two complex numbers can be easily remembered in worded form.

THEOREM 1. *The absolute value of the product of two complex numbers is equal to the product of their absolute values. The amplitude is equal to the sum of the amplitudes.*

This law of multiplication may be applied repeatedly to give the product of three or more complex numbers. Thus for three numbers we have

$$\begin{aligned}
&r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)\cdot r_3(\cos\theta_3 + i\sin\theta_3)\\
&\qquad\qquad = r_1r_2r_3[\cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)].
\end{aligned}$$

To find the quotient of $r_1(\cos\theta_1 + i\sin\theta_1)$ divided by $r_2(\cos\theta_2 + i\sin\theta_2)$, we multiply the dividend and divisor by $(\cos\theta_2 - i\sin\theta_2)$. Thus we have

$$\begin{aligned}
\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\\
&= \frac{r_1[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}\\
&= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].
\end{aligned}$$

We state this formula for the quotient of two complex numbers in the following verbal form.

THEOREM 2. *The absolute value of the quotient of two complex numbers is the quotient of their absolute values. The amplitude is the amplitude of the dividend minus the amplitude of the divisor.*

The product and quotient of two complex numbers can be obtained graphically. To find the product of the numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$, draw the angle $(\theta_1 + \theta_2)$ in standard position and measure off a distance along the terminal side equal to r_1r_2. To find the quotient of $r_1(\cos\theta_1 + i\sin\theta_1)$ divided by $r_2(\cos\theta_2 + i\sin\theta_2)$, draw the angle $(\theta_1 - \theta_2)$ in standard position and measure off a distance of $r_1 \div r_2$ along the terminal side.

Example 1. Express the numbers $-1 + i\sqrt{3}$ and $3 - 3i$ in polar form and find their product.

Solution. We have for the polar form

$$-1 + i\sqrt{3} = 2(\cos 120^\circ + i\sin 120^\circ)$$

and

$$3 - 3i = 3\sqrt{2}(\cos 315^\circ + i\sin 315^\circ).$$

Hence,

$$\begin{aligned}(-1 + i\sqrt{3})(3 - 3i) &= 6\sqrt{2}(\cos 435^\circ + i\sin 435^\circ)\\ &= 6\sqrt{2}(\cos 75^\circ + i\sin 75^\circ).\end{aligned}$$

Example 2. Find the quotient of $12(\cos 30^\circ + i\sin 30^\circ)$ divided by $3(\cos 70^\circ + i\sin 70^\circ)$.

Solution. Applying the formula for division, we obtain

$$\begin{aligned}\frac{12(\cos 30^\circ + i\sin 30^\circ)}{3(\cos 70^\circ + i\sin 70^\circ)} &= 4[\cos(-40^\circ) + i\sin(-40^\circ)]\\ &= 4(\cos 320^\circ + i\sin 320^\circ).\end{aligned}$$

Exercise 84

Perform the indicated operations and express each result in rectangular form:

1. $2(\cos 30^\circ + i\sin 30^\circ)\cdot 6(\cos 60^\circ + i\sin 60^\circ)$.
2. $4(\cos 60^\circ + i\sin 60^\circ)\cdot 3(\cos 120^\circ + i\sin 120^\circ)$.
3. $3(\cos 40^\circ + i\sin 40^\circ)\cdot 2(\cos 50^\circ + i\sin 50^\circ)$.
4. $7(\cos 110^\circ + i\sin 110^\circ)\cdot(\cos 10^\circ + i\sin 10^\circ)$.
5. $6(\cos 175^\circ + i\sin 175^\circ) \div 2(\cos 25^\circ + i\sin 25^\circ)$.
6. $10(\cos 90^\circ + i\sin 90^\circ) \div 5(\cos 0^\circ + i\sin 0^\circ)$.
7. $8(\cos 125^\circ + i\sin 125^\circ) \div 4(\cos 35^\circ + i\sin 35^\circ)$.

8. $9(\cos 231° + i \sin 231°) \div (\cos 51° + i \sin 51°)$.
9. $(\cos 314° + i \sin 314°) \div (\cos 269° + i \sin 269°)$.
10. $20(\cos 10° + i \sin 10°) \div 4(\cos 70° + i \sin 70°)$.
11. $4(\cos 20° + i \sin 20°) \cdot 3(\cos 340° + i \sin 340°)$.
12. $13(\cos 185° + i \sin 185°) \cdot 2(\cos 145° + i \sin 145°)$.
13. $\frac{1}{2}(\cos 20° + i \sin 20°) \cdot 2(\cos 30° + i \sin 30°)$.
14. $(\cos 90° + i \sin 90°) \div (\cos 50° + i \sin 50°)$.

Express the numbers in polar form and then perform the indicated operations:

15. $(1 - i)(1 + i\sqrt{3})$.
16. $(\sqrt{3} + i)(1 + i)$.
17. $(\sqrt{3} + i)(3 - 3i)$.
18. $(3 - 3i)(4 - 4i)$.
19. $(2\sqrt{3} + 2i)(1 + i\sqrt{3})$.
20. $4i(5 + 5i)$.
21. $3i(1 - i\sqrt{3})(\sqrt{3} + i)$.
22. $(2 + 2i)(\sqrt{3} - i)(1 - i\sqrt{3})$.
23. $\dfrac{1 + i}{1 + i\sqrt{3}}$.
24. $\dfrac{2\sqrt{3} - 2i}{1 - i\sqrt{3}}$.
25. $\dfrac{3}{2 - 2i}$.
26. $\dfrac{4}{3\sqrt{3} - 3i}$.
27. $\dfrac{3 + 3i}{5i}$.
28. $\dfrac{4 + 4i\sqrt{3}}{2i}$.

126. De Moivre's Theorem. By direct application of the law of multiplication of complex numbers we obtain

$$[r(\cos \theta + i \sin \theta)]^2 = r^2(\cos 2\theta + i \sin 2\theta)$$

and

$$[r(\cos \theta + i \sin \theta)]^3 = r^3(\cos 3\theta + i \sin 3\theta).$$

These results suggest that the multiplication may be continued and that, for any positive integer n, we will have the formula

$$\boldsymbol{[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).} \tag{1}$$

This formula, known as **De Moivre's theorem,** has been proved to hold not only for positive integral values of n but for all real values of n. Omitting the proof, we shall apply the formula where n is a positive integer or the reciprocal of a positive integer. That is, we shall find positive integral powers and roots of complex numbers.

To find the nth roots of $r(\cos \theta + i \sin \theta)$, we replace n by $1/n$ in formula (1) and have

$$[r(\cos \theta + i \sin \theta)]^{1/n} = \sqrt[n]{r}\left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n}\right).$$

If n is taken as a positive integer and $\sqrt[n]{r}$ is restricted to the positive (principal) root, it appears that there is only one nth root. For integral values of k, however, we may write

$$r(\cos \theta + i \sin \theta) = r[\cos (\theta + k \cdot 360°) + i \sin (\theta + k \cdot 360°)],$$

and hence

$$[r(\cos\theta + i\sin\theta)]^{1/n} = \sqrt[n]{r}\left(\cos\frac{\theta + k\cdot 360^\circ}{n} + i\sin\frac{\theta + k\cdot 360^\circ}{n}\right). \qquad (2)$$

The right member of this equation may take n distinct values by assigning to k the values $0, 1, 2, \cdots, n-1$. Any other integral value of k yields a repetition and not a new root. Thus there are n different nth roots of any nonzero number. Also, it may be observed that the amplitudes of the roots are such that graphically the roots are equally spaced about the circumference of a circle of radius $\sqrt[n]{r}$.

Example 1. Find the eighth power of the number $(1 + i\sqrt{3})$.

Solution. Changing to polar form and then using formula (1), we obtain

$$\begin{aligned}(1 + i\sqrt{3})^8 &= [2(\cos 60^\circ + i\sin 60^\circ)]^8\\ &= 2^8(\cos 480^\circ + i\sin 480^\circ)\\ &= 256(\cos 120^\circ + i\sin 120^\circ)\\ &= -128 + 128i\sqrt{3}.\end{aligned}$$

Example 2. Find the cube roots of -8.

Solution. We express -8 in polar form and use formula (2). Thus

$$-8 = 8[\cos(180^\circ + k\cdot 360^\circ) + i\sin(180^\circ + k\cdot 360^\circ)]$$

and

$$(-8)^{1/3} = 2\left(\cos\frac{180^\circ + k\cdot 360^\circ}{3} + i\sin\frac{180^\circ + k\cdot 360^\circ}{3}\right).$$

Giving k the values 0, 1, and 2 in succession, we get

$$(-8)^{1/3} = \begin{cases}2(\cos 60^\circ + i\sin 60^\circ) = 1 + i\sqrt{3},\\ 2(\cos 180^\circ + i\sin 180^\circ) = -2,\\ 2(\cos 300^\circ + i\sin 300^\circ) = 1 - i\sqrt{3}.\end{cases}$$

The numbers $1 + i\sqrt{3}$, -2, and $1 - i\sqrt{3}$ are the rectangular forms of the cube roots of -8. These are all the cube roots of -8; any other integral value for k gives a repetition of one of these roots. In particular, if the values 3, 4, and 5 are assigned to k, the roots are repeated in the order just mentioned.

Example 3. Find the five fifth roots of $-16 + 16i\sqrt{3}$.

Solution. Changing the number to polar form, we have

$$-16 + 16i\sqrt{3} = 32[\cos(120^\circ + k\cdot 360^\circ) + i\sin(120^\circ + k\cdot 360^\circ)].$$

Then

$$(-16 + 16i\sqrt{3})^{1/5} = 2\left(\cos\frac{120^\circ + k\cdot 360^\circ}{5} + i\sin\frac{120^\circ + k\cdot 360^\circ}{5}\right).$$

We assign k the values 0, 1, 2, 3, and 4, in succession and obtain the desired fifth roots. Thus,

$$(-16 + 16i\sqrt{3})^{1/5} = \begin{cases} 2(\cos 24^\circ + i\sin 24^\circ), \\ 2(\cos 96^\circ + i\sin 96^\circ), \\ 2(\cos 168^\circ + i\sin 168^\circ), \\ 2(\cos 240^\circ + i\sin 240^\circ), \\ 2(\cos 312^\circ + i\sin 312^\circ). \end{cases}$$

Each of these fifth roots $-16 + 16i\sqrt{3}$ could be changed to rectangular form by finding the values of the trigonometric functions involved. We show the roots plotted in Figure 32.

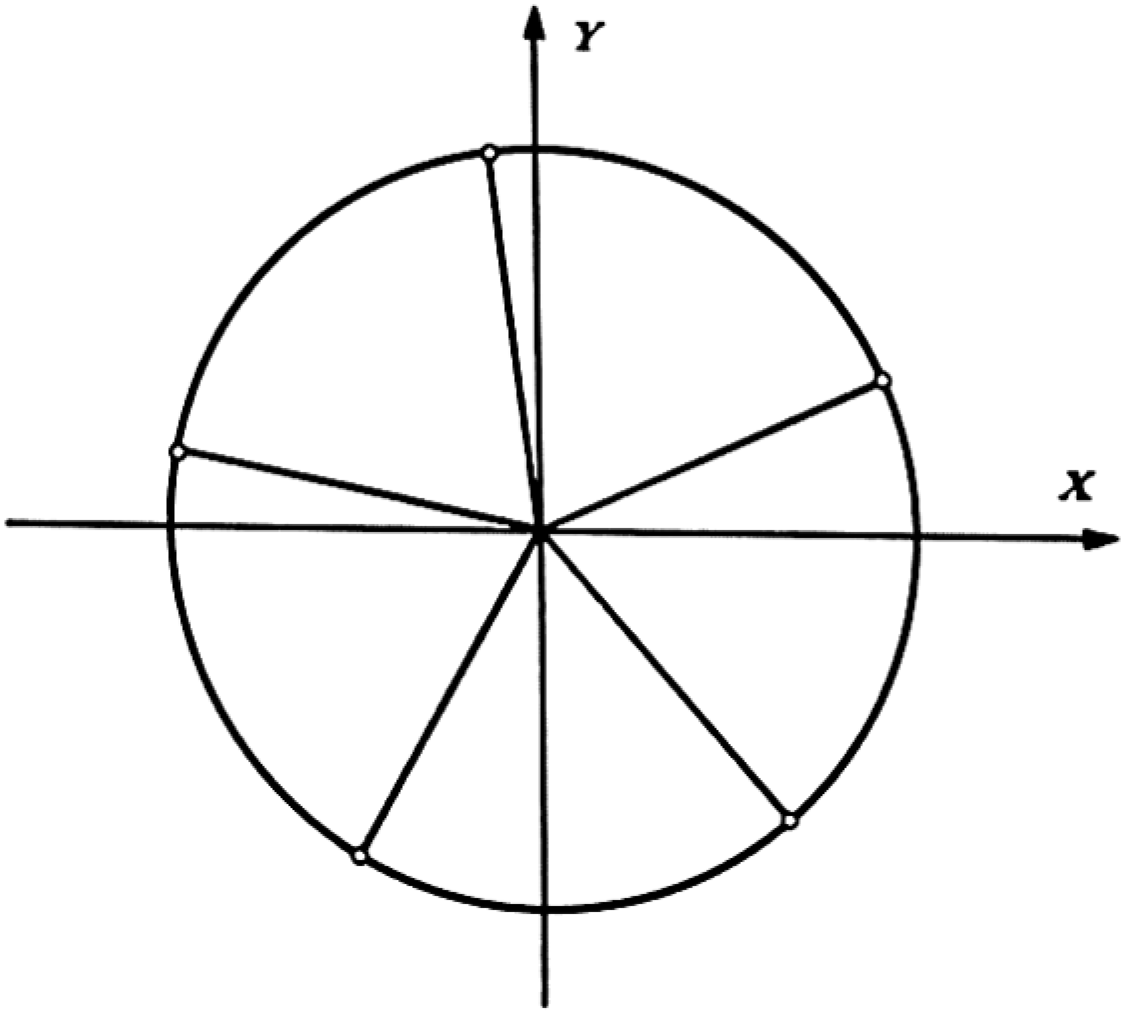


FIG. 32

Exercise 85

Find the indicated powers in problems **1–14** by use of De Moivre's theorem. Leave the results in polar form.

1. $[3(\cos 20^\circ + i\sin 20^\circ)]^2$.
2. $[2(\cos 35^\circ + i\sin 35^\circ)]^3$.
3. $[2(\cos 118^\circ + i\sin 118^\circ)]^3$.
4. $[5(\cos 220^\circ + i\sin 220^\circ)]^2$.
5. $[4(\cos 50^\circ + i\sin 50^\circ)]^4$.
6. $(\cos 10^\circ + i\sin 10^\circ)^8$.
7. $(\cos 70^\circ + i\sin 70^\circ)^{10}$.
8. $[2(\cos 300^\circ + i\sin 300^\circ)]^7$.

9. $(1 + i)^5$.
10. $(1 - i)^6$.
11. $(\sqrt{3} - i)^4$.
12. $(1 + \sqrt{3})^6$.
13. $(-\sqrt{3} + i)^{10}$.
14. $(2 - 2i)^5$.

Find the indicated roots in each problem **15–26.** Leave the results in polar form. Plot the roots of each number.

15. $[16(\cos 80° + i \sin 80°)]^{1/2}$.
16. $[25(\cos 134° + i \sin 134°)]^{1/2}$.
17. $[27(\cos 150° + i \sin 150°)]^{1/3}$.
18. $[81(\cos 160° + i \sin 160°)]^{1/4}$.
19. $(\cos 100° + i \sin 100°)^{1/5}$.
20. $[16(\cos 280° + i \sin 280°)]^{1/4}$.

21. $(1 + i)^{1/3}$.
22. $i^{1/3}$.
23. $(-i)^{1/4}$.
24. $(-625)^{1/4}$.
25. $(81)^{1/4}$.
26. $(-27)^{1/3}$.

Find the indicated powers and roots, giving each result in rectangular form:

27. $[2(\cos 50° + i \sin 50°)]^6$.
28. $[3(\cos 36° + i \sin 36°)]^5$.
29. $(\cos 210° + i \sin 210°)^4$.
30. $(\cos 120° + i \sin 120°)^4$.

31. $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^{10}$.
32. $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^{20}$.
33. $\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^{8}$.
34. $(5 + i)^{1/5}$.
35. $(3 - 2i)^{1/3}$.
36. $(1 + i\sqrt{3})^{1/4}$.

Solve the following equations:

37. $x^4 + 1 = 0$.
38. $x^6 - 1 = 0$.
39. $x^4 - i = 0$.

40. $x^2 + (1 - i)x - i = 0$.
41. $x^2 - 2x + (1 + i) = 0$.

Chapter Seventeen

PARTIAL FRACTIONS

127. Introduction. We have combined fractions into a single fraction by the process of addition. In certain mathematical situations, particularly in calculus, it is necessary to perform the inverse operation; that is, to express a fraction as the sum of simpler fractions. The resulting fractions are called **partial fractions.**

We shall consider fractions in which the numerator and denominator of each are polynomials in one variable. A fraction of this kind is called a **proper fraction** if the degree of the numerator is less than the degree of the denominator. Otherwise, the fraction is an improper fraction. An improper fraction can be expressed as the sum of a rational integral part and a proper fraction by dividing the numerator by the denominator. Hence, in finding partial fractions, we need to consider proper fractions only. Accordingly our problem in this chapter is that of separating, or resolving, proper fractions into partial fractions.

Illustrations:

$$\frac{x^2 + 23x - 18}{(2x - 1)(x + 2)(x - 1)} \text{ is a proper fraction.}$$

$$\frac{2x^3 - 5x^2 - 3x + 10}{x^2 - 1} \text{ is an improper fraction.}$$

The first of these fractions is the sum of three fractions. The second is the sum of a rational integral part and two proper fractions. These statements may be verified by adding the quantities in the right members of the following equations.

$$\frac{x^2 + 23x - 18}{(2x - 1)(x + 2)(x - 1)} = \frac{5}{2x - 1} - \frac{4}{x + 2} + \frac{2}{x - 1}.$$

$$\frac{2x^3 - 5x^2 - 3x + 10}{x^2 - 1} = 2x - 5 + \frac{2}{x - 1} - \frac{3}{x + 1}.$$

In adding fractions a common denominator is used which has each of the separate denominators as a factor. Conversely, one might conclude that the denominators of the partial fractions are factors of the denominator of the given fraction. Although this conclusion is correct, it does not fully suggest a procedure for our problem. The process of finding partial fractions is based on a theorem whose proof is beyond our present stage. Hence we shall assume the following statements concerning a proper fraction in its lowest terms:

1. Corresponding to each linear factor $ax + b$ *of the denominator there will be a partial fraction*

$$\frac{A}{ax + b},$$

where A *is a constant.*

2. Corresponding to each repeated linear factor $(ax + b)^k$ *of the denominator there will be* k *partial fractions*

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}.$$

where $A_1, A_2, \cdots, A_k$ *are constants and* $A_k \neq 0$.

3. If $ax^2 + bx + c$ *is a factor of the denominator, and not the product of two real linear factors, then corresponding to this quadratic factor there will be a partial fraction*

$$\frac{Ax + B}{ax^2 + bx + c},$$

where A *and* B *are constants.*

4. If $ax^2 + bx + c$ *is not the product of two real linear factors, then corresponding to the repeated factor* $(ax^2 + bx + c)^k$ *of the denominator there will be* k *partial fractions*

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k},$$

where $A_1, B_1, A_2, B_2, \cdots, A_k, B_k$ *are constants with* A_k *and* B_k *not both zero.*

We shall show illustrative examples for each of the four types of factors mentioned in the assumptions.

128. Case 1. We first illustrate the case in which the factors of the denominator are linear with no factor repeated.

Example. Separate $\dfrac{x^2 + 23x - 18}{(x - 1)(x + 2)(2x - 1)}$ into partial fractions.

Solution by Substitution. In accordance with item 1 the partial fractions have the denominators $x - 1$, $x + 2$, $2x - 1$, and their numerators are constants. Hence we write

$$\frac{x^2 + 23x - 18}{(x - 1)(x + 2)(2x - 1)} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{2x - 1}. \tag{1}$$

The constants A, B, and C are to be determined so that this equation is an identity. To facilitate their determination we clear equation (1) of fractions and have

$$\begin{aligned} &x^2 + 23x - 18 \\ &\quad = A(x + 2)(2x - 1) + B(x - 1)(2x - 1) + C(x - 1)(x + 2). \end{aligned} \tag{2}$$

This equation, also an identity, must be true for any value which is assigned to x (Sec. 113, Theorem 4). The most convenient values are 1, -2, and $\frac{1}{2}$. The constants are readily found by substituting each of these values in turn in equation (2).

For $x = 1$: $\quad 1 + 23 - 18 = A(3)(1)$ and $A = 2$.

For $x = -2$: $\quad 4 - 46 - 18 = B(-3)(-5)$ and $B = -4$.

For $x = \frac{1}{2}$: $\quad \frac{1}{4} + \frac{23}{2} - 18 = C(-\frac{1}{2})(\frac{5}{2})$ and $C = 5$.

Therefore,

$$\frac{x^2 + 23x - 18}{(x - 1)(x + 2)(2x - 1)} = \frac{2}{x - 1} - \frac{4}{x + 2} + \frac{5}{2x - 1}.$$

Solution by Equating Coefficients. We may evaluate the constants A, B, and C in equation (2) by using the fact that the coefficients of like powers of x in the two members of the equation are equal. To follow this plan we perform the indicated multiplications and collect like terms. Thus we obtain

$$\begin{aligned} &x^2 + 23x - 18 \\ &\quad = (2A + 2B + C)x^2 + (3A - 3B + C)x + (-2A + B - 2C). \end{aligned}$$

Equating the coefficients, we obtain

$$\begin{aligned} 2A + 2B + C &= 1, \text{ from coefficients of } x^2, \\ 3A - 3B + C &= 23, \text{ from coefficients of } x, \\ -2A + B - 2C &= -18, \text{ from the constant terms.} \end{aligned}$$

Solving this system for A, B, and C, we find, as before, $A = 2$, $B = -4$, $C = 5$.

Either the method of substitution or the method of equating coefficients of the like powers of the variable may be used to determine the unknown constants. The substitution method works particularly well when the factors of the denominator are linear. Frequently, however, it is advantageous to combine the two methods.

Exercise 86

Resolve into partial fractions and check by addition:

1. $\dfrac{x-5}{(x+1)(x-1)}$.
2. $\dfrac{3}{x(x+3)}$.
3. $\dfrac{7x-16}{(x-2)(x-3)}$.
4. $\dfrac{7x-8}{(2x-1)(x-2)}$.
5. $\dfrac{3x-7}{(3x+1)(3x-1)}$.
6. $\dfrac{5}{(x+1)(2x-3)}$.
7. $\dfrac{11x+1}{3x^2-2x-1}$.
8. $\dfrac{3x+1}{2x^2-x-6}$.
9. $\dfrac{22-4x}{3x^2+2x-8}$.
10. $\dfrac{6x+27}{4x^3-9x}$.
11. $\dfrac{8x-8}{x^3-2x^2-8x}$.
12. $\dfrac{5x-2}{6x^3-5x^2+x}$.
13. $\dfrac{4x^2-27x+20}{(x-3)(x-4)(x+2)}$.
14. $\dfrac{2x+8}{(x+2)(x-2)(x+1)}$.
15. $\dfrac{15}{(2x+1)(3x-1)(4x-1)}$.
16. $\dfrac{3x^2-15x+10}{(1-x)(2-x)(3-x)}$.

129. Case 2. When the denominator is the product of linear factors with one or more repeated, we assign denominators to the partial fractions in accordance with item 2, Section 127.

Example. Resolve $\dfrac{4x^2-13x}{(x+3)(x-2)^2}$ into partial fractions.

Solution by Substitution. This fraction is the sum of partial fractions with $x+3$, $x-2$, and $(x-2)^2$ as denominators and with constant numerators. Hence,

$$\frac{4x^2-13x}{(x+3)(x-2)^2}=\frac{A}{x+3}+\frac{B}{x-2}+\frac{C}{(x-2)^2}. \tag{1}$$

Clearing of fractions, we obtain

$$4x^2-13x=A(x-2)^2+B(x+3)(x-2)+C(x+3). \tag{2}$$

We substitute successively the values -3, 2, and 0 for x in this equation.

For $x=-3$: $36+39=25A$ and $A=3$.

For $x=2$: $16-26=5C$ and $C=-2$.

For $x = 0$:
$$0 = 4A - 6B + 3C,$$
$$6B = 4A + 3C = 12 - 6 = 6,$$
$$B = 1.$$

Hence,
$$\frac{4x^2 - 13x}{(x+3)(x-2)^2} = \frac{3}{x+3} + \frac{1}{x-2} - \frac{2}{(x-2)^2}.$$

Solution by Equating Coefficients. To determine the constants A, B, and C by equating the coefficients of like powers of x, we perform the multiplications indicated in equation (2) and collect to get
$$4x^2 - 13x = (A + B)x^2 + (-4A + B + C)x + (4A - 6B + 3C).$$

Hence, from the coefficients of x^2, x, and the constant terms, we obtain
$$A + B = 4$$
$$-4A + B + C = -13$$
$$4A - 6B + 3C = 0.$$

The solution of this system of equations is $A = 3$, $B = 1$, $C = -2$, which agrees with the previous determination.

Exercise 87

Resolve into partial fractions:

1. $\dfrac{x-5}{(x-4)^2}$.
2. $\dfrac{7-8x}{(2x-1)^2}$.
3. $\dfrac{2x+7}{(x+3)^2}$.
4. $\dfrac{x^2-x-3}{(x-1)^3}$.
5. $\dfrac{18x^2+3x+4}{(3x+1)^3}$.
6. $\dfrac{x^2-2x+3}{(2-x)^3}$.
7. $\dfrac{2x^2-2x-1}{(x-2)(x-1)^2}$.
8. $\dfrac{6x^2-11x-32}{(x+6)(x+1)^2}$.
9. $\dfrac{6x+3}{(x-1)^2(x-4)}$.
10. $\dfrac{5x^2+36x-27}{x^4-6x^3+9x^2}$.
11. $\dfrac{x^3-3x^2+x-3}{(x+1)(x-1)^3}$.
12. $\dfrac{x^2+5x-3}{2x^4-3x^3}$.
13. $\dfrac{7x^3+2x^2-27x+14}{(3x+2)(2x-3)(x-2)^2}$.
14. $\dfrac{7x^3+32x^2-15x-32}{(2x-1)(3x+1)(x+3)^2}$.
15. $\dfrac{2x^3+2x^2+2x+4}{x(x+1)(x+2)^2}$.
16. $\dfrac{6x^3-15x^2+9x+3}{(2x-1)^2(x-2)^2}$.

130. Case 3. When the denominator contains a quadratic factor which is not the product of two real linear factors the quadratic factor is the denominator of one of the partial fractions. A linear expression is provided for the numerator of a quadratic denominator (Sec. 127, item 3).

Example 1. Resolve $\dfrac{x^3 - 25x^2 + 21x - 45}{(x - 3)(x + 2)(2x^2 - x + 3)}$ into partial fractions.

Solution. The quadratic factor $2x^2 - x + 3$ is not the product of two real linear factors. Hence, the partial fractions will have $x - 3$, $x + 2$, and $2x^2 - x + 3$ as denominators. The numerators of the linear denominators are constants, but the numerator of the quadratic denominator may be a linear expression. Thus we have

$$\frac{x^3 - 25x^2 + 21x - 45}{(x - 3)(x + 2)(2x^2 - x + 3)} = \frac{A}{x - 3} + \frac{B}{x + 2} + \frac{Cx + D}{2x^2 - x + 3}. \qquad (1)$$

Clearing of fractions, we obtain

$$\begin{aligned} x^3 - 25x^2 + 21x - 45 &= A(x + 2)(2x^2 - x + 3) \\ &\quad + B(x - 3)(2x^2 - x + 3) + (Cx + D)(x - 3)(x + 2). \qquad (2) \end{aligned}$$

To determine the constants A, B, C, D, we shall combine the method of substitution and the method of equating coefficients. We first substitute the values 3, -2, and 0 for x in equation (2).

$$\begin{aligned} &\text{For } x = 3: & -180 &= A(5)(18) \quad \text{and} \quad A = -2. \\ &\text{For } x = -2: & -195 &= B(-5)(13) \quad \text{and} \quad B = 3. \\ &\text{For } x = 0: & -45 &= 6A - 9B - 6D, \\ & & -45 &= -12 - 27 - 6D \quad \text{and} \quad D = 1. \end{aligned}$$

Equating coefficients of x^3 gives

$$\begin{aligned} 1 &= 2A + 2B + C, \\ 1 &= -4 + 6 + C \quad \text{and} \quad C = -1. \end{aligned}$$

Hence,

$$\frac{x^3 - 25x^2 + 21x - 45}{(x - 3)(x + 2)(2x^2 - x + 3)} = \frac{-2}{x - 3} + \frac{3}{x + 2} + \frac{1 - x}{2x^2 - x + 3}.$$

Example 2. Resolve $\dfrac{4x^3 - 10x^2 - 2x - 9}{(2x^2 - x + 3)(3x^2 + x + 1)}$ into partial fractions.

Solution.

$$\frac{4x^3 - 10x^2 - 2x - 9}{(2x^2 - x + 3)(3x^2 + x + 1)} = \frac{Ax + B}{2x^2 - x + 3} + \frac{Cx + D}{3x^2 + x + 1}. \qquad (1)$$

Clearing of fractions gives

$$\begin{aligned} 4x^3 &- 10x^2 - 2x - 9 \\ &= (Ax + B)(3x^2 + x + 1) + (Cx + D)(2x^2 - x + 3). \qquad (2) \end{aligned}$$

To obtain four equations from (2), involving the unknowns A, B, C, and D, let us equate the coefficients of x^3 and then substitute in turn the values 0, 1, and -1 for x.

Equate coefficients of x^3: $\quad 4 = 3A + 2C.$

For $x = 0$: $\quad -9 = B + 3D.$

For $x = 1$: $\quad -17 = 5A + 5B + 4C + 4D.$

For $x = -1$: $\quad -21 = -3A + 3B - 6C + 6D.$

From the first two equations of this system, we have

$$C = \frac{4 - 3A}{2} \quad \text{and} \quad B = -9 - 3D.$$

Then substituting these results in the other two equations, and solving for A and D, we get

$$A = 2, \quad D = -2,$$

and finally

$$B = 3, \quad C = -1.$$

Therefore,

$$\frac{4x^3 - 10x^2 - 2x - 9}{(2x^2 - x + 3)(3x^2 + x + 1)} = \frac{2x - 3}{2x^2 - x + 3} - \frac{x + 2}{3x^2 + x + 1}.$$

Exercise 88

Resolve into partial fractions:

1. $\dfrac{x^2 + 3x - 1}{(x - 2)(x^2 + 5)}$.

2. $\dfrac{x^2 - x + 2}{(x + 1)(x^2 + 3)}$.

3. $\dfrac{x^2 - 5x - 3}{(x + 3)(x^2 - 3x + 3)}$.

4. $\dfrac{1 - 13x - 10x^2}{(x - 2)(2x^2 + 2x + 1)}$.

5. $\dfrac{x^2 + 6x - 12}{(x - 4)(x^2 - x - 5)}$.

6. $\dfrac{x^2 + 6x - 12}{(x + 2)(x^2 + x + 2)}$.

7. $\dfrac{3x^3 + 8x^2 + 6x - 2}{(x + 2)(x + 1)(x^2 + 2)}$.

8. $\dfrac{4x^3 + x^2 + 14x - 4}{x(x - 1)(x^2 + 4)}$.

9. $\dfrac{x^2 - 7x + 1}{(x + 1)^2(x^2 - x + 1)}$.

10. $\dfrac{3x^3 - 8x^2 + x - 2}{(x - 1)^2(x^2 + x + 1)}$.

11. $\dfrac{x^3 + 5x^2 + 4x + 7}{(x^2 + 2)(x^2 + 1)}$.

12. $\dfrac{2x^3 + 3x^2 + 3x + 2}{(3x^2 + 2)(2x^2 + 3)}$.

13. $\dfrac{x^3 + x^2 - 2x + 1}{(2x^2 + x + 2)(3x^2 - x + 1)}$.

14. $\dfrac{3x^3 - 4x^2 + 10x + 4}{(x^2 + 2x + 2)(x^2 - 2x + 2)}$.

131. Case 4. When the denominator of the given fraction has a repeated quadratic factor we form the denominators of the corresponding partial fractions as in the case of a repeated linear factor. The numerators, of course, are to be linear expressions.

Example. Resolve $\dfrac{x^3 + 4x^2 + 5x + 3}{(x + 1)(x^2 + x + 1)^2}$ into partial fractions.

Solution. We use $x + 1$, $x^2 + x + 1$, and $(x^2 + x + 1)^2$ as the denominators of the partial fractions. The numerator of $x + 1$ is a constant; the other numerators may be linear expressions. Hence,

$$\frac{x^3 + 4x^2 + 5x + 3}{(x + 1)(x^2 + x + 1)^2} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + x + 1} + \frac{Dx + E}{(x^2 + x + 1)^2}. \qquad (1)$$

Clearing (1) of fractions gives

$$x^3 + 4x^2 + 5x + 3 = A(x^2 + x + 1)^2 + (Bx + C)(x + 1)(x^2 + x + 1) + (Dx + E)(x + 1). \qquad (2)$$

Substituting -1 for x in (2), we obtain at once $A = 1$. To get equations for finding the remaining constants, we next perform the multiplications in the right member of (2). We have, after putting $A = 1$,

$$x^3 + 4x^2 + 5x + 3 = x^4 + 2x^3 + 3x^2 + 2x + 1 + B(x^4 + 2x^3 + 2x^2 + x) + C(x^3 + 2x^2 + 2x + 1) + D(x^2 + x) + E(x + 1).$$

Collecting gives

$$x^3 + 4x^2 + 5x + 3 = (1 + B)x^4 + (2 + 2B + C)x^3 + (3 + 2B + 2C + D)x^2 + (2 + B + 2C + D + E)x + (1 + C + E).$$

Equating coefficients, we obtain

$1 + B = 0$, from coefficients of x^4,

$2 + 2B + C = 1$, from coefficients of x^3,

$3 + 2B + 2C + D = 4$, from coefficients of x^2,

$1 + C + E = 3$, from the constant terms.

The first two of these equations give readily $B = -1$ and $C = 1$. Then the last two yield $E = 1$ and $D = 1$. Hence,

$$\frac{x^3 + 4x^2 + 5x + 3}{(x + 1)(x^2 + x + 1)^2} = \frac{1}{x + 1} - \frac{x - 1}{x^2 + x + 1} + \frac{x + 1}{(x^2 + x + 1)^2}.$$

Exercise 89

Resolve into partial fractions:

1. $\dfrac{3x^3 + 4x - 5}{(x^2 + 2)^2}$.

2. $\dfrac{x^3 + x^2 - x + 4}{(x^2 + 1)^2}$.

3. $\dfrac{3x^2 + 5x + 3}{(x^2 + x + 1)^2}$.

4. $\dfrac{x^3 + x^2 + 4x + 1}{(x^2 + 2)^3}$.

5. $\dfrac{x^4 - 2x^3 + 9x^2 - 11x + 21}{(x^2 - x + 4)^3}$.

6. $\dfrac{2x^3 - x^2 + 6x - 11}{(x^2 + x + 3)^3}$.

7. $\dfrac{x^4 + 4x^2 - x + 3}{(x^2 + 1)^3}$.

8. $\dfrac{3x^4 - 13x^3 + 30x^2 - 36x + 27}{x(x^2 - 2x + 3)^2}$.

9. $\dfrac{5x^2 + 18}{x(x^2 + 3)^2}$.

10. $\dfrac{x^3 - 2x^2 + 1}{x^2(2x^2 + 1)^2}$.

11. $\dfrac{x^5 + 11x^3 + 10x^2 + 25x + 50}{x^2(x^2 + 5)^2}$.

12. $\dfrac{x^4 + 4x^2 - x + 2}{(x + 2)(x^2 + 2)^2}$.

13. $\dfrac{x^4 - 6x^3 + 9x^2 - 10x + 5}{(x - 3)(x^2 - 2x + 2)^2}$.

14. $\dfrac{4x^4 + 7x^3 + 18x^2 - 3x + 11}{(2x - 1)(x^2 + x + 2)^2}$.

Chapter Eighteen

PERMUTATIONS AND COMBINATIONS

132. Introduction. Permutations and combinations, which we shall define and study in this chapter, have a strong recreational appeal to many people. But aside from this happy interest, they have important practical and theoretical applications. As a first step, we shall examine a principle which is fundamental in the study of these topics.

An illustration will be helpful in understanding the principle. Suppose a governor is to be chosen from three candidates and a lieutenant governor from four other candidates. Let us determine the number of different ways the two positions can be filled. There are three ways of filling the office of governor and with each of these possible ways there are four choices for a lieutenant governor. Hence, in all, there are $3 \cdot 4 = 12$ different ways of making the two choices. In order to check this result we designate the candidates for governor by A, B, and C and the candidates for lieutenant governor by x, y, z, and w. Then the positions can be filled in any of the following 12 ways:

$$\begin{array}{cccc} A\,w & A\,x & A\,y & A\,z \\ B\,w & B\,x & B\,y & B\,z \\ C\,w & C\,x & C\,y & C\,z \end{array}$$

By generalizing the reasoning in this illustration we may make the following statement:

FUNDAMENTAL PRINCIPLE. *If a thing can be done in any of m different ways and then a second thing can be done in any of n different ways, it follows that the total number of different ways in which the first and then the second thing can be done is m times n.*

The truth of this principle is easily seen since for each of the m ways of doing the first thing there are n ways of doing the second. It is also equally evident that the principle may be extended to three or more things.

Example 1. How many two-digit numbers can be made from the digits 1, 2, 3, 4, 5 if (a) no digit is repeated in a number; (b) repetitions are allowed?

Solution. (a) There are five choices for the tens' digit, and after a choice is made there remain four choices for the units' digit. Hence the total possibilities are $5 \cdot 4 = 20$. The same result, of course, would be obtained by first selecting the units' digit and then the tens' digit.

(b) If repetitions are allowed there are five choices for the tens' digit and five choices for the units' digit, making $5 \cdot 5$, or 25 possible two-digit numbers.

Example 2. In how many ways can five people sit in a car, two in the front seat and three in the back seat, if a particular two of the five are to avoid the driver's position?

Solution. Let us designate the five positions by the letters A, B, C, D, E with A indicating the driver's seat. Taking these places in the order named, let us write above each letter the number which tells how many ways it can be filled.

$$\begin{matrix} 3 & 4 & 3 & 2 & 1 \\ A & B & C & D & E \end{matrix}$$

We have three choices for a driver, four choices for place B, three choices for place C, two choices for place D, and one choice for place E. Hence the total number of seating arrangements is $3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 72$.

Exercise 90

1. How many two-digit numbers can be made with the digits 2, 4, 6, 8 if (a) repetitions are allowed; (b) no digit is to be repeated in a number?

2. How many three-digit numbers can be formed with the digits 1, 2, 3, 4, 5 if (a) repetitions are allowed; (b) no repetitions are allowed?

3. How many three-digit numbers (zero not the first digit) can be formed with digits 0, 1, 2, 3, 4 if (a) repetitions are allowed; (b) no repetitions are allowed?

4. How many four-digit numbers, each less than 5000, can be formed with the digits 1, 2, 4, 6, 8 if (a) repetitions are allowed; (b) no repetitions are allowed?

5. How many committees consisting of 1 freshman, 1 sophomore, and 1 junior can be selected from 40 freshmen, 30 sophomores, and 25 juniors?

6. Five trails lead to a hilltop. In how many ways can one ascend and descend by (a) a different trail; (b) any trail?

7. Five boys are in a room which has 4 doors. In how many different ways can they leave the room?

8. A building has 6 outside doors. In how many ways can one enter and leave (a) by a different door; (b) by any door?

9. How many signals can be made with 5 different kinds of flags placed on a flagpole one under another from the highest? How many signals can be made if only 4 of the flags are used at a time?

10. License plates are to be distinguished by using 2 letters followed by four digits. How many different plates are possible (a) if letters and digits may be repeated; (b) if no repetitions are allowed?

11. In how many ways can 3 people be seated in a room where there are 7 seats?

12. In how many ways can 5 books be arranged on a shelf? How many ways if a certain 2 are not to be separated?

13. Four couples are to eat at a round table with the men and women alternating. If she reserves a place for herself, in how many ways can the hostess assign seats to the others?

14. How many seating arrangements are possible in problem **13** if the host is not to sit by the hostess?

15. A club has 21 members. In how many ways can a president and a vice-president be elected from the group if no member is eligible for both positions?

16. In how many ways can 5 men and 4 women be seated in a row of 9 seats (a) with the men and women alternating; (b) if the women are not to be separated; (c) if the women are to be together and the men together?

17. Towns A and B are connected by 3 roads and towns B and C by 4 roads. In how many ways can one drive from A to B to C and return to A via B if (a) the same road is not to be traveled twice; (b) the same road may be traveled twice?

18. In how many ways can a club of 21 members choose a president, a secretary, and a treasurer if no member may hold more than one of the positions?

19. In how many ways can 9 books, 6 with green covers and 3 with red covers, be arranged on a shelf if (a) the red books are not to be separated; (b) the green books are not to be separated; (c) books of the same color are kept together?

20. How many odd, three-digit numbers (zero not the first digit) can be formed with the digits 0, 1, 2, 3, 4, 5, 6 if (a) no digit is repeated in a number; (b) repetitions are allowed?

133. Permutations. An arrangement of a group of things in a definite order is called a **permutation.** The letters a, b, and c, for example, have the following possible arrangements, or permutations:

$$a\ b\ c \qquad a\ c\ b \qquad b\ a\ c \qquad b\ c\ a \qquad c\ a\ b \qquad c\ b\ a$$

As a further illustration, we write the permutations of the digits 1, 2, and 3, where only two are used at a time.

$$12 \qquad 13 \qquad 21 \qquad 23 \qquad 31 \qquad 32$$

In each of these illustrations we have written the six permutations. It is easy to determine the number of permutations, however, without actually writing them. To find the number of permutations which can be made with a, b, and c, we notice that there are three positions to be filled. Hence there are three choices for the first position, two for the second position, and one for the third position. According to the fundamental principle, there are $3 \cdot 2 \cdot 1$ different ways of writing the three letters. In the case of the digits only two were used at a time. Hence there are three choices for the first position and two for the second, making a total of six possible permutations.

Let us now apply this reasoning to the case of n different things where r of the things (r a part or all) are used at a time. We may regard this problem as one in which selections are to be made from n things to fill r positions. There are n choices for filling the first position, and then $n - 1$ choices for the second position, $n - 2$ choices for filling the third position, and so on. We continue this process until the rth position is filled, and notice that there are $n - r + 1$ choices for this position. Recalling the fundamental principle, we see that the number of permutations of n things taken r at a time is given by the product of the r integers from n to $n - r + 1$, inclusive. Denoting this product by $P(n, r)$, we write the formula

$$\mathbf{P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1).} \tag{1}$$

This formula may be expressed in a more compact form by multiplying and dividing by $(n - r)!$. Thus we get

$$P(n, r) = \frac{n(n - 1)(n - 2) \cdots (n - r + 1)(n - r)!}{(n - r)!}$$

and therefore

$$\mathbf{P(n, r) = \frac{n!}{(n - r)!}}. \tag{2}$$

By putting $r = n$ in formula (1), we obtain the number of permutations of n things taken all at a time. This gives the product of the integers from 1 to n, inclusive; and hence

$$\mathbf{P(n, n) = n!.}$$

Example. How many permutations can be made from the letters in the word MONDAY if (a) 4 letters are used at a time; (b) all letters are used; (c) all letters are used but the first is a vowel?

Solution. (a) We want to find the number of permutations of 6 things taken 4 at a time. Using either formula (1) or the fundamental principle, we have

$$P(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360.$$

(b) From formula (3), or returning to the fundamental principle, we have

$$P(6, 6) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

(c) The first letter in each permutation is to be either "*o*" or "*a*." Therefore we have two choices for the first letter. But after selecting the first letter, we can select the remaining 5 without restriction, and consequently in 5! ways. The total number of ways of selecting the first letter and then the others is $2 \cdot 5! = 240$.

134. Permutations of Things Not All Different. The formulas of Section 133 were derived on the assumption that the set of n things, or objects, includes objects that are all different. The formulas do not apply if some of the objects are alike. The word ALASKA, for example, has six letters. But we cannot make 6! permutations using the six letters at a time because three letters are alike. To aid in finding the number of different permutations, let us assign subscripts to the three A's. Thus writing

$$A_1 \ L \ A_2 \ S \ K \ A_3,$$

we have six distinct objects. Having distinguished the A's, we now have 6! possible permutations. Letting L, S, and K remain in fixed positions, we permute the A's in their positions. This can be done in 3! ways as shown.

$$\begin{array}{lll} A_1 \ L \ A_2 \ S \ K \ A_3 & A_1 \ L \ A_3 \ S \ K \ A_2 & A_2 \ L \ A_1 \ S \ K \ A_3 \\ A_2 \ L \ A_3 \ S \ K \ A_1 & A_3 \ L \ A_1 \ S \ K \ A_2 & A_3 \ L \ A_2 \ S \ K \ A_1 \end{array}$$

These six arrangements would be indistinguishable if the subscripts were removed. Thus we can see that any permutation of the six letters can be replaced by 3! permutations by making the A's distinct. Hence the number of permutations P of 6 objects with 3 alike when multiplied by 3! gives the number of permutations of 6 different objects. That is,

$$3!P = 6! \quad \text{and} \quad P = \frac{6!}{3!} = 120.$$

This reasoning can be applied to the general case of n objects of which n_1 are identical. In this case any arrangement of the n objects could be replaced by $n_1!$ permutations by making the n_1 objects distinct. Hence we write the following theorem.

THEOREM. *The number of permutations P of n things taken all at a time, where n_1 of the objects are alike and the others distinct, is*

$$P = \frac{n!}{n_1!}.$$

This formula may be extended to the cases where there are two, three, or more groups of like *objects* among the n objects. Thus we write

$$P = \frac{n!}{n_1!n_2!n_3! \cdots}.$$

Example 1. How many permutations can be made with the 9 letters in the word TENNESSEE?

Solution. There are 2 S's, 2 N's, and 4 E's, and therefore

$$P = \frac{9!}{2!2!4!} = 3780.$$

Example 2. A man tosses 1 half dollar, 2 quarters, 3 dimes, and 4 nickels among 10 boys. In how many different ways can the boys profit if each is to get a coin?

Solution. If the coins were all different there would be 10! ways in which each boy could get a coin. But since there are groups of one, two, three, and four like coins, we have

$$P = \frac{10!}{2!3!4!} = 12{,}600.$$

135. Circular Permutations. In our discussion thus far we have supposed that a permutation is formed by placing one object after another in a line. Thus there is a first and last object. If the objects are arranged in a circle, however, there is no first or last object, and the formulas of Section 133 are not directly applicable. A circular arrangement presents a new aspect because a mere rotation would not change the position of any object relative to the others. Hence a rotation does not constitute a different permutation.

To find the number of ways of arranging n different objects in a circle, we first select a position for one of the objects. Then the others can be placed in their positions in $(n - 1)!$ different ways. Six people, for example, can form a circle in 5! ways. They could, of course, form a line in 6! ways.

Example. In how many ways can four couples be seated at a round table with the men and women alternating?

Solution. We first think of assigning one, say a woman, to a chair. This fixes 3 particular places for the other women and 4 particular places for the men. The women, therefore, can be assigned to their places in 3! ways and the men in 4! ways. Applying the fundamental principle, we get the result $3!4! = 144$.

Exercise 91

1. Evaluate (a) $P(7, 1)$; (b) $P(7, 3)$; (c) $P(7, 4)$; (d) $P(7, 7)$.

2. Evaluate $P(4, 1) + P(4, 2) + P(4, 3) + P(4, 4)$.

3. Show that (a) $P(5, 4) = P(5, 5)$; (b) $P(6, 5) = P(6, 6)$; (c) $P(n, n - 1) = P(n, n)$.

4. How many ways can the letters of the word TEXAS be arranged if (a) all letters are used; (b) 2 are used at a time; (c) 3 are used at a time?

5. Find the number of permutations, each with 7 letters, which can be made with the letters in (a) WYOMING; (b) KENTUCKY; (c) WASHINGTON.

6. How many three-digit numbers can be made with the digits 1, 2, 3, 4, 5, 6 if no digit is repeated in a number? How many four-digit numbers?

7. Find how many even numbers of three-digits each can be made with the digits 1, 2, 3, 4, 6, 7 if no digit is repeated.

8. How many five-digit numbers, each less than 70,000, can be made with the digits 1, 2, 5, 7, 9 if no digit is repeated?

9. Seven songs are to be given in a program. In how many different orders could they be rendered?

10. In how many ways can 6 people be seated in a row of 9 chairs?

11. In how many ways can 8 books be arranged on a shelf if a certain 3 are (a) not to be separated; (b) not to be separated and left in a particular order?

12. Find the number of possible batting orders for a baseball team of 9 players if the pitcher is to bat last.

13. Five different mathematics books, 4 different physics books, and 2 different history books are to be placed on a shelf with the books of each subject together. Find the number of ways in which the books can be placed.

14. How many permutations can be made by using all the letters in (a) OHIO; (b) INDIANA; (c) ASAMAYAMA?

15. Find the number of permutations which can be formed by using all the letters in (a) ALABAMA; (b) ARKANSAS; (c) MISSISSIPPI.

16. How many signals can be made by arranging 9 flags in a line if 4 are red, 3 are white, and 2 are blue?

17. Five copies of a mathematics book, 4 copies of a physics book, and 4 copies of a history book are to be placed on a shelf. If the covers are of different colors for each kind of book, how many noticeably different ways can the books be placed on the shelf?

18. In how many ways is it possible for 10 customers to buy 5 cans of peaches, 3 cans of pineapple, and 2 cans of tomatoes if each customer gets 1 can?

19. Find the number of ways 2 half dollars, 4 quarters, and 6 dimes can be distributed among 12 boys if each gets a coin?

20. Seven children join hands and form a circle. In how many ways could the circle be formed? In how many ways could they make a line?

21. Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line. How many different ways could they form a circle in which the boys and girls alternate?

22. Ten people are to sit at a round table. Find the number of seating arrangements if the host and hostess are to sit opposite each other.

23. In how many ways can 8 persons be seated at a round table if a certain 2 are (a) to sit next to each other; (b) not to sit next to each other?

24. Four couples are to be seated at a round table. Find the number of seating arrangements if each couple is not to be separated.

136. Combinations. A part or the entirety of a group of objects is called a **combination** of the group. A combination, in *distinction* from a permutation, has no *reference to order*. For example, in a committee selected from a group of people, our interest is in the individuals rather than the order of selection or other arrangment of the committeemen. If all of a group of objects are taken at a time there is, of course, only one combination. By taking a part of the objects at a time, however, there are more possible combinations than one. We shall be interested in learning how to determine the number of combinations for this situation.

The letters a, b, c, d taken 3 at a time have the following combinations:

$$a\ b\ c \qquad a\ b\ d \qquad a\ c\ d \qquad b\ c\ d$$

Changing the order of one of these combinations does not make a new combination. Thus, $a\ b\ c$ and $b\ c\ a$ are two permutations of the same things, or of the same combination.

Let $C(n, r)$ stand for the number of combinations of n things taken r at a time, where r is a positive integer less than n. Now each combination of the r things can be permuted in $r!$ ways. But if each of the combinations is thus permuted, then all possible permutations of n things taken r at a time are obtained. Hence, returning to formula (1), Section 133, we have

$$r!C(n, r) = n(n-1)(n-2)\cdots(n-r+1),$$

and

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{1}$$

Using formula (2), Section 133, we have the form

$$C(n, r) = \frac{n!}{r!(n-r)!}. \tag{2}$$

It may be observed that these formulas hold not only for $r < n$ but also for $r = n$. Substituting n for r in formulas (1) and (2), we get

$$C(n, n) = \frac{n!}{n!} = 1 \quad \text{and} \quad C(n, n) = \frac{n!}{n!0!} = 1.$$

We use here the customary definition of factorial zero ($0! = 1$).

When r things are taken from a group of n things, $n - r$ things are left. Thus we see that for each combination of r things there is a corresponding combination of $n - r$ things. We conclude, therefore, that the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n - r$ at a time. This conclusion can be reached also by replacing r by $n - r$ in formula (2) above. This gives

$$C(n, n-r) = \frac{n!}{[n-(n-r)]!(n-r)!} = \frac{n!}{r!(n-r)!}.$$

Hence,

$$C(n, r) = C(n, n-r).$$

Let us now examine the coefficients in the expansion of $(a + x)^n$ when n is a positive integer (Sec. 103).

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3$$

$$+\cdots+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots + x^n.$$

The coefficients of this expansion, starting with the coefficient n of the second term, are the combinations of n things taken 1 at a time, 2 at a time, 3 at a time, and so on, to n at a time. That is, referring to formula (1), we see that the coefficients are

$$C(n, 1), C(n, 2), C(n, 3), \cdots, C(n, r), \cdots, C(n, n).$$

Writing the coefficients in this way, we have

$$(a + x)^n = a^n + C(n, 1)a^{n-1}x + C(n, 2)a^{n-2}x^2 + C(n, 3)a^{n-3}x^3 + \cdots$$

$$+ \cdots C(n, r)a^{n-r}x^r + \cdots + C(n, n)x^n.$$

We replace a by 1 and x by 1 in this identity and obtain

$$(1 + 1)^n = 1 + C(n, 1) + C(n, 2) + C(n, 3) + \cdots + C(n, n),$$

and, consequently,

$$\mathbf{C(n, 1) + C(n, 2) + C(n, 3) + \cdots + C(n, n) = 2^n - 1.}$$

We state this result as a theorem.

THEOREM. *The total number of combinations of n things taken successively* 1 *at a time,* 2 *at a time,* 3 *at a time, and so on, to n at a time is* $2^n - 1$.

Example 1. In how many ways can a committee be selected from 18 persons if the committee is to have (a) 3 members; (b) 14 members?

Solution. (a) We wish to find the number of combinations of 18 things taken 3 at a time. Hence, by formula (1),

$$C(18, 3) = \frac{18 \cdot 17 \cdot 16}{3!} = 816.$$

(b) We may substitute in formula (1) and evaluate $C(18, 14)$, but it is easier to recall that $C(18, 14) = C(18, 4)$ and write

$$C(18, 4) = \frac{18 \cdot 17 \cdot 16 \cdot 15}{1 \cdot 2 \cdot 3 \cdot 4} = 3060.$$

Example 2. In how many ways can 7 mathematics teachers be employed from 10 men applicants and 7 lady applicants if (*a*) 3 are to be men; (b) 3 or 4 are to be men?

Solution. (a) The 3 men can be had in $C(10, 3)$ different ways. The remaining 4 can be had from the lady applicants in $C(7, 4)$ ways. Hence the total number of ways is

$$C(10, 3) \cdot C(7, 4) = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \cdot \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 4200.$$

(b) To find the number of ways of filling the vacancies with either 3 or 4 men among those employed, we add the number of ways of filling the vacancies with 3 men and 4 ladies to the number of ways of employing 4 men and 3 ladies. That is, we have

$$C(10, 3) \cdot C(7, 4) + C(10, 4) \cdot C(7, 3) = 4200 + 7350 = 11{,}550.$$

Example 3. Find how many different committees consisting of 1, 2, 3, 4, or 5 members can be formed from 5 people.

Solution. We need to find the number of combinations of 5 things taken one or more at a time. Substituting 5 for n in the formula $2^n - 1$, we have $2^5 - 1 = 31$ as the number of different committees.

Exercise 92

1. Evaluate (a) $C(31, 2)$; (b) $C(26, 3)$; (c) $C(24, 4)$; (d) $C(13, 5)$.

2. Taking full advantage of possible cancellations, evaluate $C(16, 4)$ and $C(16, 12)$ by formula (1). Evaluate both by use of formula (2).

3. How many different ways can a tennis team of 4 be chosen from 17 players?

4. Twelve people meet in a room and each shakes hands with all the others. Find the number of handshakes.

5. Nine points, no three of which are on a straight line, are marked on a blackboard. How many lines, each through 2 of the points, can be drawn? How many triangles are determined by the points?

6. Find the number of choices which a student has for doing 6 problems from a set of 11 problems.

7. Seven different coins are tossed simultaneously. In how many ways can 3 heads and 4 tails come up?

8. In how many ways can a committee of 5 be chosen from a club of 12 members if (a) the president must be on the committee; (b) the president is not eligible; (c) a certain two members refuse to serve together on the committee?

9. In how many ways can a court of 9 judges make (a) a 5 to 4 decision; (b) a majority decision?

10. A presiding officer is authorized to appoint a committee of 5 from 4 elders and 8 deacons. How many different committees are possible if 3 elders are to be members?

11. A committee of 7 is to be composed of 4 Republicans and 3 Democrats. Find the number of choices if 7 Republicans and 8 Democrats are available for appointment.

12. A grocery store manager wishes to employ 6 checkers from 4 men applicants and 5 lady applicants. Find the number of ways of employing the 6 if (a) 3 are to be men; (b) 2 or 3 are to be men.

13. How many different groups of 9 baseball players can be chosen from 3 who play only as pitchers, 2 who play only as catchers, and 14 who play in any position other than pitcher or catcher?

14. Find the number of ways of selecting 8 books from 10 different mathematics books and 7 different physics books if (a) 5 are to be mathematics books; (b) 3 or 4 are to be physics books.

15. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls, and 4 blue balls if each selection consists of 3 balls of each color.

16. A committee of 9 is to be chosen from 10 seniors, 8 juniors, and 7 sophomores. Find the number of choices if the committee is to have 4 seniors, 3 juniors, and 2 sophomores.

17. In how many ways can 7 pennies lie horizontally on a table so that one or more heads are up?

18. How many sums of money can be formed with a penny, a nickel, a dime, a quarter, and a half dollar?

19. In how many ways can a person invite one or more of 10 friends to dinner?

20. Find how many sums of money, each consisting of 3 or more coins, can be formed from 6 different kinds of coins?

21. In how many ways can a group of 3 or more be chosen from 8 people?

22. Find the number of ways in which a librarian can check out 8 different kinds of books to two students so that each gets 4 books. Suggestion: Find the number of ways of giving 4 books to the first student.

23. In how many ways can 6 different kinds of books be distributed among three children if each child is to have 2 books?

24. In how many ways can 12 people be separated into two groups of 5 and 7?

25. Find the number of ways in which 11 people can be separated into two groups of 5 and 6 if a certain one of the 12 is to be in the smaller group.

Chapter Nineteen

PROBABILITY

137. Introduction. Probability is a fascinating subject, and it has held the interest of mathematicians to the extent that its theory has been highly developed during the past two or three centuries. Receiving its first impetus from games of chance, today probability has wide and important applications in insurance, annuities, and the social and physical sciences. Many people who have made no formal mathematical or philosophical study of probability have some understanding of its concepts and applications.

We shall consider some elementary aspects of probability, and make applications to familiar situations. The principles which we shall take up, however, are fundamental to the more advanced theory.

138. Probability in Single Events. If an event can happen in m ways and can fail to happen in n ways, and if each of these $m + n$ ways is equally likely, then in a single trial producing one of the results the probability that the event will happen is defined by

$$p = \frac{m}{m + n},$$

and the probability of failure is

$$q = \frac{n}{m + n}.$$

For example, if a ball is drawn at random from a bag containing 3 white balls and 4 red balls, there are 3 ways in which a white ball can be drawn and 4 ways in which a white ball will not be drawn. Then the probability of getting a white ball is $\frac{3}{7}$ and the probability of getting a red ball is $\frac{4}{7}$.

If the m ways in which an event can happen is greater than the n ways in which it can fail, then the odds are said to be m to n in favor of success (i.e. of its happening); but for $m < n$, the odds are n to m against the event.

It should be noticed that $p + q = 1$, and this means that the event will either happen or fail to happen in a trial. For a situation in which either p or q is 0, the other is equal to 1. If all the balls in a bag are white, the probability of drawing a white ball is 1 and the probability of drawing a ball not white is 0.

There are many events which may be analyzed to determine the number of ways for success and the number of ways for failure. Probabilities thus determined are called **mathematical** probabilities.

Example 1. Six pennies are tossed simultaneously. What is the probability that (a) all are heads; (b) 3 are heads and 3 are tails?

Solution. (a) There are two ways for a penny to fall, heads up or tails up. By the fundamental principle (Sec. 132), the six pennies can fall in $2^6 = 64$ ways. All heads can occur in only 1 way. Therefore the probability of obtaining all heads is $\frac{1}{64}$.

(b) A given set of 3 pennies can fall heads up in 1 way. But there are $C(6, 3) = 20$ ways of getting sets of 3 pennies from 6 pennies. Thus we find $\frac{20}{64} = \frac{5}{16}$ as the probability for 3 heads and 3 tails.

Example 2. The names of 5 boys and 4 girls are written on cards. If 6 of the cards are drawn at random, what is the probability of getting the names of 3 boys and the names of 3 girls?

Solution. There are $C(5, 3)$ ways of drawing the names of 3 boys, and $C(4, 3)$ ways of getting the names of 3 girls. Hence, $C(5, 3) \cdot C(4, 3)$ is the number of ways for success. But the total number of ways of drawing 6 names is $C(9, 6)$. Thus,

$$p = \frac{C(5, 3) \cdot C(4, 3)}{C(9, 6)} = \frac{40}{84} = \frac{10}{21}.$$

Exercise 93

1. What is the probability that in a single throw (a) a coin will fall heads; (b) a die will fall a six?

2. Find the probability that a domino drawn from a double six set is (a) the double six; (b) a six; (c) a domino whose spots add to six?

3. Eight white balls and 2 black balls are in a box. What is the probability that in drawing 1 ball it is (a) black; (b) white?

4. If a single die is cast, find the probability that it falls (a) a one or two; (b) a four, five, or six?

5. Two pennies are tossed simultaneously. Find the probability that (a) both fall heads; (b) one falls heads and the other tails.

6. In a throw of 2 dice, find the probability that (a) both are aces; (b) the sum of the spots is 7; (c) just one is an ace.

7. Two dominoes are drawn from a double six set. Find the probability that (a) both are doubles; (b) just one is a double; (c) both are sixes.

8. Five pennies are tossed simultaneously. Find the probability that (a) all are heads; (b) 4 are heads and 1 is tails; (c) 2 are heads and 3 are tails.

9. A store has 7 doors at an entrance. What is the probability that a customer using the entrance will enter and leave by (a) a particular door; (b) the same door; (c) different doors?

10. In an examination, the questions and answers are given but are not paired. A student succeeds in pairing all but 5 questions and 5 answers. These he does wholly by guessing. What is the probability that he pairs (a) all of them correctly; (b) only 3 correctly?

11. Eight children, 4 boys and 4 girls, draw for places to form a line. Find the probability that (a) the boys will be together in the line; (b) a particular couple will be together.

12. Three balls are drawn from a bag containing 6 red balls and 5 black balls. Find the probability that (a) all are red; (b) all are black; (c) 2 are red and 1 is black.

13. If two cards are drawn from a deck of 52 cards, what is the probability that both are (a) queens; (b) hearts; (c) of the same suit?

14. A baby in playing with the wooden digits 1, 2, 3, 4, 5 places them in a row. What is the probability that the number thus formed is (a) less than 20,000; (b) more than 40,000?

15. At a party, 8 men draw for partners from 8 cards which have the names of their wives. What is the probability that each man will draw the name of his wife?

16. Three balls are drawn from a bag containing 6 green balls, 7 black balls, and 8 yellow balls. Find the probability that (a) they are of different colors; (b) all of the same color.

17. Ten people, 5 boys and 5 girls, are grouped into pairs as partners. If the pairs are formed by drawing, find the probability that each boy has a girl for his partner.

18. A party of 5 men and 5 women are to eat at a round table. If the hostess reserves a place for herself and the others take places at random, find the probability that the men and women alternate.

19. Three cards are drawn from a deck of 52 cards. Find the probability that they are (a) hearts; (b) of the same suit; (c) of different suits.

20. A single card is drawn from each of three decks of 52 cards. Find the probability that the cards thus drawn are (a) hearts; (b) of the same suit; (c) of different suits.

21. Eight books, 5 red and 3 green, are taken from a table one after the other at random. Find the probability that (a) all the red books are taken first; (b) all the books of one color are taken first.

139. Probability in Multiple Events. If in two or more events the happening of one does not affect the probability of the happening of any of the others, the events are said to be **independent.** For example, the event of casting a die a first time and the event of casting the die a second time are independent. The result of the first throw has no bearing on the face which comes up on the second throw. We have the following theorem relating to independent events.

THEOREM 1. *The probability that two or more independent events will happen is given by the product of their separate probabilities.*

Proof: Suppose that the first event can happen in m_1 ways out of a total of M_1 ways of happening or failing to happen. Then the probability of the event is $p_1 = m_1/M_1$. Similarly, suppose that $p_2 = m_2/M_2$ is the probability of the second event. By the fundamental principle of Section 132, both events can happen in m_1m_2 ways out of a total of M_1M_2 ways of happening or failing to happen. Then the probability that both events will happen is

$$\frac{m_1m_2}{M_1M_2} = \frac{m_1}{M_1} \cdot \frac{m_2}{M_2} = p_1p_2.$$

The proof can be extended easily to more than two independent events.

Example 1. A man casts a die twice. Find the probability of (a) an ace on both throws; (b) an ace only on the first throw.

Solution. (a) Since a die has six faces, the probability of an ace on the first throw is $\frac{1}{6}$, and the probability of an ace on the second throw is likewise $\frac{1}{6}$. The two throws are independent events, and therefore the probability of two aces is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

(b) The probability of an ace on the first throw is $\frac{1}{6}$, and the probability of a number other than the ace on the second throw is $\frac{5}{6}$. Hence the probability that both events will happen is $\frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$.

Two events are said to be **dependent** if the happening of one affects the probability that the other will happen. For example, if one ball is drawn and then another is drawn from a bag of red and blue balls, the probability that the second ball will be blue, say, depends on the color of the ball which was first removed. We state a theorem relating to dependent events.

THEOREM 2. *If p_1 is the probability that an event will happen and, after it has happened, p_2 is the probability for a second event, then the probability that the first event and then the second event will happen is p_1p_2.*

The proof for this case and its extension to more dependent events may be made in just the same manner as for independent events.

Example 2. A ball is drawn from a box containing 6 red balls and 7 blue balls and is not returned. Then a second ball is drawn. Find the probability that (a) both balls are red; (b) the first is blue and the second red.

Solution. (a) The probability of getting a red ball on the first draw is $\frac{6}{13}$. If a red ball is drawn, 12 balls remain, 5 of which are red. This makes the probability of a red ball on the second draw $\frac{5}{12}$. Hence the probability that both balls are red is $\frac{6}{13} \cdot \frac{5}{12} = \frac{5}{26}$.

(b) The probability of a blue ball on the first draw is $\frac{7}{13}$, and the probability of a red ball on the second draw is $\frac{6}{12}$. The product $\frac{7}{13} \cdot \frac{6}{12} = \frac{7}{26}$ is the probability of getting first a blue ball and then a red ball.

Two or more events are said to be **mutally exclusive** if it is impossible for more than one of them to happen in a single trial. For example, in drawing one ball from a bag of red and white balls, the event of getting a red ball and the event of getting a white ball are mutually exclusive. Only one of the events is possible in a single trial. We state and prove a theorem relating to mutually exclusive events.

THEOREM 3. *The probability that some one of two or more mutually exclusive events will happen is the sum of the probabilities of the separate events.*

Proof: Let n stand for the number of possible results in a single trial, and m_1 denote the number of ways one event can happen in the trial and m_2 the number of ways another event can happen. Since the two events are mutually exclusive, the m_1 and m_2 ways are all different. This means that out of a total of n possibilities there are $m_1 + m_2$ ways in which one or the other event can happen. Hence the probability p that one of the events will happen is

$$p = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = p_1 + p_2.$$

The proof may be readily extended to more than two mutually exclusive events.

Example 3. If 2 balls are drawn from a box containing 5 red balls and 7 blue balls, what is the probability that both are of the same color?

Solution. The event of getting 2 red balls and the event of getting 2 blue balls are mutually exclusive. The probability of getting 2 red balls (using the method of Section 138) is

$$p_1 = \frac{C(5, 2)}{C(12, 2)} = \frac{5}{33}.$$

The probability of getting 2 blue balls is

$$p_2 = \frac{C(7, 2)}{C(12, 2)} = \frac{7}{22}.$$

Hence the probability of getting 2 balls of the same color is $\frac{5}{33} + \frac{7}{22} = \frac{31}{66}$.

Instead of using the method of Section 138, the drawing of 2 balls could be regarded as 2 dependent events. Accordingly, the probability of getting a red ball on the first draw is $\frac{5}{12}$, and after a red ball is drawn the probability for a second red ball is $\frac{4}{11}$. Hence, the probability for 2 red balls is $\frac{5}{12} \cdot \frac{4}{11} = \frac{5}{33}$. Similarly, it is evident that the probability of drawing 2 blue balls is $\frac{7}{12} \cdot \frac{6}{11} = \frac{7}{22}$.

If the probability of success in a single trial is p, then in n trials the **expected number** of successes is np. In 200 throws of a coin the expected number of heads is $\frac{1}{2}(200) = 100$. In 300 throws of a die the expected number of aces is $\frac{1}{6}(300) = 50$.

If the probability that a person will receive a sum of A dollars is p, then the value of his **mathematical expectation** is pA dollars.

Example 4. A man holds 8 tickets in a drawing for a prize of $150. If there are a total of 1000 tickets, what is the value of his expectation?

Solution. The man's probability of winning is $\frac{8}{1000}$. Then the value of his expectation by multiplication is $1.20.

Exercise 94

1. A ball is drawn from a box containing 4 red balls and 5 green balls. It is then returned to the box and a second ball is drawn. Find the probability that (a) both balls are red; (b) both are green; (c) the first is red and the second green.

2. Repeat problem **1** if the first ball is not returned.

3. If a coin is tossed three times in succession, find the probability that it falls (a) heads each time; (b) heads first and tails on the other throws.

4. One card is drawn from each of three decks of 52 cards. What is the probability that (a) all are aces; (b) all are face cards; (c) all are spades?

5. Three cards are drawn from a deck one at a time and are not replaced. What is the probability that (a) all are aces; (b) all are face cards; (c) all are spades?

6. If the probability that a brush salesman will make a sale at one house is $\frac{2}{3}$ and the probability that he will make a sale at a second house is $\frac{1}{2}$, find the probability that he will make (a) both sales; (b) neither sale; (c) a sale at the first house but not at the second house.

7. A store entrance has 4 doors. What is the probability that the next three customers, coming singly, will enter by (a) the same door; (b) different doors?

8. A committee of 3 is to be formed by lot from a group of 24 students. If the group is made up of 9 freshmen, 8 sophomores, and 7 juniors, find the probability that the committee will have as members (a) 3 freshmen; (b) one from each class; (c) all from one class.

9. A student is to take 3 tests. If the separate probabilities of his passing the tests are $\frac{4}{5}$, $\frac{3}{4}$, and $\frac{2}{3}$, find the probability that he will pass (a) all tests; (b) none of the tests; (c) exactly two of the tests.

10. To be initiated into an organization a candidate must pass at least two of three tests. If the separate probabilities of his passing the tests are $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{4}{5}$, find the probability that he may win the right to be initiated.

11. Three balls are drawn from a box one at a time and are not replaced. If the box had 5 white balls and 7 green balls, find the probability that the 3 balls are drawn in the order (a) white, green, green; (b) white, white, green; (c) green, green, green.

12. Three balls are drawn from a box one at a time and are not replaced. If the box had 5 red balls, 6 white balls, and 7 blue balls, find the probability that the three drawn balls are of the same color.

13. The names of 6 girls and 5 boys are drawn from a box. Find the probability that (a) the girls names are drawn first; (b) the names of the girls and boys alternate in the drawing.

14. A man has three entries in a horse race. He figures that the separate probabilities for his horses to be first are $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$. On this basis, find the probability that one of his horses wins. If there is a prize of $500 for the winner, what is the value of the man's expectation?

15. To determine which will do a task, three men match coins on the basis that the odd man is to do the task. Find the probability that the decision will be made on (a) the first trial; (b) the second trial.

16. Repeat problem **15** if the odd man is relieved of the task and the others then match to determine the issue.

17. The separate probabilities that marksmen A, B, and C will hit a target are $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$. Shooting in the order A, B, and C, the first to hit the target gets a prize of $100, but not more than a total of 3 shots are allowed. Find the value of each man's mathematical expectation.

18. A, B, and C are to decide on one of them to do a task. A and B first draw lots, and the loser then draws with C. Find each man's probability of having to do the task.

19. A person draws 3 balls in succession from a box containing 5 red balls, 6 yellow balls, and 7 green balls. He is to receive a prize of $100 if the balls are drawn in the order red, yellow, and green. Find the value of his mathematical expectation if (a) each ball is replaced before the next is drawn; (b) if the balls are not replaced.

20. Find the value of the man's expectation in problem **19** if a second chance is granted in case of failure in the first attempt.

140. Probabilities in Repeated Trials. Suppose n trials are made of an event in which the probability of a success in a single trial is p and of failure q. Let us find the probabilities of exactly 0, 1, 2, 3, and so on, successes in the n trials. The separate trials are, of course, independent events. The probability of no success, or of n failures is q^n. The probability of a success on a specified trial and failure on all the others is $q^{n-1}p$. But the one success could happen on any of the n trials, or in $C(n, 1)$ ways. Hence, for exactly one success, the probability is $C(n, 1)q^{n-1}p$. Now the probability of successes on two specified trials and failures on all the others is $q^{n-2}p^2$. The two successes, however, could occur in $C(n, 2)$ different ways among the n trials. Therefore, the probability of exactly two successes is $C(n, 2)q^{n-2}p^2$. Similarly, for exactly three successes the probability is $C(n, 3)q^{n-3}p^3$. Hence, for the general case of r successes and $n - r$ failures, the probability is $C(n, r)q^{n-r}p^r$. But these probabilities are terms of the binomial expansion

$$(q + p)^n = q^n + C(n, 1)q^{n-1}p + C(n, 2)q^{n-2}p^2 + \cdots + C(n, r)q^{n-r}p^r + \cdots + p^n.$$

The first term of this expansion gives the probability of no success in n trials; the second term is the probability of exactly one success; the third term is the probability of exactly two successes, and so on. We may now state the following theorem

THEOREM. *If p is the probability of success and q the probability of failure in a single trial, then the probability of exactly r successes in n trials is*

$$C(n, r)p^rq^{n-r}.$$

Example. In 4 throws of a die, what is the probability of (a) exactly 2 aces; (b) 2 or more aces?

Solution. (a) The probability of an ace on a single throw is $\frac{1}{6}$, and of failure $\frac{5}{6}$. Hence the required probability is

$$C(n, r)p^rq^{n-r} = C(4, 2)(\tfrac{1}{6})^2(\tfrac{5}{6})^2 = \tfrac{25}{216}.$$

(b) The probability of 2, 3, or 4 aces is given by the sum of these mutually exclusive events. Hence we have

$$C(4, 2)(\tfrac{1}{6})^2(\tfrac{5}{6})^2 + C(4, 3)(\tfrac{1}{6})^3(\tfrac{5}{6}) + (\tfrac{1}{6})^4 = 150(\tfrac{1}{6})^4 + 20(\tfrac{1}{6})^4 + (\tfrac{1}{6})^4$$
$$= \frac{171}{6^4} = \frac{171}{1296}.$$

141. Probability from Experiment. We have discussed probabilities arrived at by analyzing the situations involved. The events have been of such nature that the number of ways of success and the number of ways of failure (and all equally likely) could be determined mathematically. There are numerous situations in which probabilities cannot be had from such logical deductions. For example, it would be impossible, by reasoning alone, to determine the probability that an untested rifleman will hit a target in a single trial. But suppose that he is allowed to shoot a great number of times, say 100, and that he makes 80 hits. With this information we might figure that his probability of hitting the target in a single trial is $\frac{80}{100}$. Obviously, however, we could not reliably estimate his shooting skill from a quite small number of trials.

If in n trials (n a large number) an event happens m times, then the probability that it will happen in a single trial is defined by the ratio m/n. Probabilities thus arrived at are called **experimental** or **statistical** probabilities.

Insurance companies make extensive use of statistical probabilities. They are interested especially in the mortality rates of people. The American Experience Mortality Table (Table III at back of book) was made from data gathered by life insurance companies. Starting with 100,000 children at age 10, this table predicts the number of deaths during the year and the number still living at the end of each year. Later studies, however, have revealed that the mortality rate of this table is somewhat too high.

Example 1. Using the American Experience Mortality Table, find the probability that a child 10 years old will live to be 40.

Solution. Referring to the table, we find that out of 100,000 children at age 10 there will be 78,106 living at the age of 40. Hence the probability that a child of 10 will be living at age 40 is

$$p = \frac{78{,}106}{100{,}000} = 0.781.$$

Example 2. A man is 40 and his wife is 38. What is the probability that (a) both will be living 10 years later; (b) only one will be living?

Solution. (a) Out of 78,106 living at age 40, there are 69,804 living at age 50; and out of 79,611 living at age 38, there are 71,627 living at age 48. Hence, the respective probabilities are

$$p_1 = \frac{69{,}804}{78{,}106} = 0.894, \quad p_2 = \frac{71{,}627}{79{,}611} = 0.900.$$

Then $p_1p_2 = (0.894)(0.900) = 0.805$, the probability that both will be living 10 years later.

(b) There are two ways in which only one will be living—the wife surviving the husband or the husband surviving the wife. The probability that the husband will not be living is $1 - 0.894 = 0.106$. Hence, the probability that only the wife will be living is

$$(0.106)(0.900) = 0.095.$$

The probability that the wife will not be living is $1 - 0.900 = 0.100$. Then the probability that only the husband will be living is

$$(0.100)(0.894) = 0.089.$$

Adding these probabilities of two mutually exclusive events, we have 0.184 as the probability that only one will be living at the end of 10 years.

Exercise 95

1. A coin is to be tossed 4 times. Find the probability that there will be (a) exactly 2 heads; (b) exactly 3 heads; (c) fewer than 2 heads.

2. Find the probability that in 5 throws of a die there will be (a) no six; (b) 2 sixes; (c) 4 or 5 sixes.

3. A box contains 4 red balls and 3 white balls. If 5 balls are drawn in succession and each is replaced before the next is taken, find the probability that (a) all are white; (b) exactly 2 are white; (c) 3 or more are red.

4. In 6 tosses of a coin, find the respective probabilities of exactly 0, 1, 2, 3, 4, 5, and 6 heads. What number of heads is most likely?

5. A ball player has a batting average of .300. Find the probability that in 4 times at bat he will (a) go hitless; (b) make exactly 1 hit; (c) make more than 1 hit.

6. A rifleman has hit a target 90 times in 100 trials. Find the probability that in 5 trials he will make (a) no hit; (b) exactly 3 hits; (c) 5 hits.

7. The probability that team A will win a basketball game from team B is 0.4. In a series of 5 games, what number of games is A most likely to win? Is A more likely to win exactly 1 or exactly 3 games?

8. Find the probability that in three times at bat a player will get exactly 2 hits if in the last 100 times at bat he has made (a) 20 hits; (b) 30 hits; (c) 40 hits.

9. Five dominoes are drawn from a double six set. If each domino is replaced before the next drawing, find the probability that (a) no double is drawn; (b) exactly 2 doubles are drawn; (c) exactly 4 doubles are drawn.

10. A bag contains 3 white balls and 1 black ball. One ball is drawn from the bag and replaced, and then another is drawn. If this is continued until 5 balls have been drawn, find the respective probabilities of drawing the black ball exactly 0, 1, 2, 3, 4, and 5 times.

11. What is the probability that a person 15 years old will live to be (a) 30; (b) 45; (c) 60?

12. Find the probability that a man 20 years old will (a) live to be 30; (b) die between 30 and 40.

13. A boy 10 years old is to receive $5000 at the age of 18. Find the value of his expectation.

14. A man is to receive $1000 when he is 40 years old. Find the value of his expectation if he is now (a) 20; (b) 30.

15. What is the probability that a child 10 years old will die during his 60th year?

16. Find the probability that a man 25 years old and his wife 20 years old will be living 20 years later. Find the probability that only one will be living.

17. Find the probability that a man 60 and his wife 55 will be living 15 years later. Find the probability that only one will be living.

18. A father, mother, and child are 40, 38, and 15 years old respectively. Find the probability that 10 years later (a) all will be living; (b) only 2 will be living.

19. The parents of two children, aged 10 and 15, are both 37 years old. Find the probability that 10 years later (a) all will be living; (b) only 3 will be living.

Chapter Twenty

DETERMINANTS

142. Introduction. Determinants of the second and third orders were defined in Section 50, and applications were made in solving systems of linear equations. Determinants, however, have numerous other uses, and are frequently employed even in elementary mathematical situations. Hence we shall give a general definition of determinants and study some of their properties.

First, however, let us explain the notion of inversions which is helpful in the definition and study of determinants. In a sequence of integers, an **inversion** occurs whenever a larger number precedes a smaller number. Thus, in the arrangement of the numbers 2, 1, 4, 3 there are two inversions—2 precedes 1, and 4 precedes 3. But 5, 1, 4, 3 has four inversions; 5 before 1, 4, and 3 furnishes three inversions, and 4 preceding 3 gives another. Inversions of letters, out of their alphabetical order, are similarly defined.

143. Definition of a Determinant. We now give a definition of a determinant which applies to those of order one, two, three or higher.

A square array of numbers forming n rows and n columns with vertical lines on either side constitutes the symbol of an nth order determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & s_1 \\ a_2 & b_2 & c_2 & \cdots & s_2 \\ a_3 & b_3 & c_3 & \cdots & s_3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & b_n & c_n & \cdots & s_n \end{vmatrix}$$

This symbol stands for a number which is the algebraic sum of all possible products obtained:

(1) By taking as factors one and only one number from each row and each column, and

(2) By giving to each product a plus or minus sign according as there is an even or odd number of inversions of the subscripts when the letters of each product are in the order of their appearance in the first row.

Each of the numbers in the square array is called an **element** of the determinant. The elements along the line from the upper left to the lower right corner form the **principal diagonal.** Each of the signed products described in the definition is called a **term** of the determinant and the sum of the terms is the **expansion,** or **value,** of the determinant. There are, of course, n factors in each term since one and only one factor is taken from each row and each column.

In Section 50 determinants of the second and third orders were defined by giving the terms of their expansions. Using the general definition, let us verify the expansion previously given for the third order determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

The subscripts in the product $a_1b_2c_3$ have no inversion. It is given a plus sign since zero is an even number. In $a_3b_1c_2$, 3 precedes 2 and 1, making two inversions and calling for a plus sign. In $a_2b_3c_1$, 2 precedes 1, and 3 precedes 1, making two inversions. The product $a_3b_2c_1$ is given a minus sign since it presents three inversions—two inversions from 3 preceding 2 and 1, and one inversion from 2 preceding 1. There is one inversion in each of the products $a_1b_3c_2$ and $a_2b_1c_3$, and consequently they take minus signs.

144. Properties of Determinants. From the definition of determinants a number of interesting and useful properties can be established.

PROPERTY I. *An nth order determinant has n! terms in its expansion.*

Proof: By writing the n letters in the order of their appearance in the first row and then using as subscripts the integers from 1 to n, inclusive, we obtain the factors of one term. But the n subscripts can, by Section 133, be assigned in $n!$ different ways. Since there is a term for each arrangement of the subscripts, we have $n!$ terms.

PROPERTY II. *The value of a determinant is not changed if corresponding rows and columns are interchanged.*

Proof: Let us make the proof for a third order determinant. We are to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

In the first determinant the subscripts reveal the row and the letters tell the column of each element. But the interchange of rows and columns reverses the roles of the subscripts and letters in the second determinant. By definition, the factors in a term are to be written in the order of their appearance in the columns, and the sign for the product is to be determined by the number of inversions of the symbols representing the rows. Following the definition then, the factors of each term of the first determinant are to be written with the letters in the order a, b, c; and each term of the second determinant is to have the subscripts in the order 1, 2, 3. The product $c_1a_2b_3$, for example, has its factors arranged properly for a term of the second determinant. There are two inversions by virtue of the c preceding a and b which means $+c_1a_2b_3$ is a term of the second determinant. Now the factors for the corresponding term of the first determinant are to be in the order $a_2b_3c_1$. To obtain this order let us first change $c_1a_2b_3$ to $a_2c_1b_3$ which introduces one inversion of the subscripts and gets rid of one inversion of the letters. Next change $a_2c_1b_3$ to $a_2b_3c_1$, introducing another inversion of the subscripts and eliminating another inversion of the letters. Hence the number of inversions of the letters in $c_1a_2b_3$ is the same as the number of inversions of the subscripts in $a_2b_3c_1$, and this product with a plus sign is a term of both determinants. Similarly, any term of one of the determinants is a term of the other.

This reasoning may be applied to a determinant of order two or greater. For interchanging two adjacent factors adds one inversion to one set of the symbols (letters or subscripts) and subtracts one from the other. So that the number of inversions of the letters where the subscripts have no inversions is the same as the number of inversions of the subscripts where the factors of the product are written in the order of the letters.

NOTE. Since the interchange of corresponding rows and columns of a determinant does not change the value of the determinant, a property of a determinant involving its rows is also a property involving its columns, and vice versa. Hence the proof of a property pertaining to rows or columns of a determinant need be made for only one of the two.

PROPERTY III. *If each of the elements of a row (or column) of a determinant is zero, then the value of the determinant is zero.*

Proof: This property follows at once from the fact that each term has a factor from every row and every column of the determinant. A product in which zero is a factor is, of course, zero.

Property IV. *If two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.*

Proof: Let us interchange two adjacent rows of a third order determinant and show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

From the first determinant the factors of $a_2b_1c_3$, for example, are in order from rows 2, 1, 3, making one inversion. The factors of this product, $a_2b_1c_3$, in the resulting determinant come in order from rows 1, 2, 3, producing no inversion of the row ordering. Hence $-a_2b_1c_3$ is a term of the first determinant and its negative, $a_2b_1c_3$, is a term of the determinant obtained by interchanging the two rows. Similarly, any term of one determinant is the negative of a term of the other. For the interchange of two adjacent rows interchanges the row order of two adjacent factors in each product of the expansion, and thus changes by one the inversions of the numbers which locate the factors by row. This statement also holds for determinants of any order greater than one. Hence the theorem is true if two adjacent rows are interchanged.

Now if k rows lie between the two rows to be interchanged, the interchange can be made by a succession of interchanges of adjacent rows. First bring the lower row next under the upper row in question by k interchanges; then interchange the two; and finally take the upper row on down through the k rows one by one. This process, involving $2k + 1$ interchanges, leaves all rows except the two in their original places. But $2k + 1$ is an odd number which means that the resulting determinant is the negative of the original one. Hence the interchange of any two rows of a determinant changes the sign of the determinant.

Property V. *If two rows (or columns) of a determinant are identical, the value of the determinant is zero.*

Proof: By Property IV, if two rows are interchanged the sign of the determinant is changed. If D is the value of the first determinant, the value of the second determinant, obtained by interchanging the identical rows, is $-D$. But the two determinants are obviously equal since all the corresponding elements are equal. Therefore $D = -D$, whence $2D = 0$, and $D = 0$.

Property VI. *If each of the elements of a row (or column) of a determinant is multiplied by a number m, the value of the determinant is multiplied by m.*

Proof: Since each term of the expansion has one and only one factor from every row, m will appear as a factor in each term of the expansion of the new determinant. Then the expansion can be written as m times the expansion of the first determinant.

Property VII. *If each element of a row (or column) of a determinant is expressed as the sum of two terms, the determinant can be expressed as the sum of two determinants.*

Proof: We shall prove that

$$\begin{vmatrix} a_1 + a'_1 & b_1 & c_1 \\ a_2 + a'_2 & b_2 & c_2 \\ a_3 + a'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}.$$

Any term of the expansion of the first determinant may be written as the sum of the corresponding terms in the expansions of the two determinants on the right. For example, $(a_1 + a'_1)b_2c_3 = a_1b_2c_3 + a'_1b_2c_3$.

Property VIII. *The value of a determinant is not changed if each element of any row (or column) is multiplied by a number m and added to the corresponding elements of any other row (or column).*

Proof: Let us show that

$$D = \begin{vmatrix} a_1 & b_1 & c_1 + ma_1 \\ a_2 & b_2 & c_2 + ma_2 \\ a_3 & b_3 & c_3 + ma_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

By Property VII, the determinant on the left may be written as the sum of two determinants. That is,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & ma_1 \\ a_2 & b_2 & ma_2 \\ a_3 & b_3 & ma_3 \end{vmatrix}.$$

But the second of these is equal to zero since it may have m factored from the elements of the third column, and written as m times a determinant with the first and third columns identical.

Exercise 96

Each of the following products is written with the factors in different orders. Determine the number of inversions of both the letters and the subscripts in each case:

1. $a_2b_1c_3$
$b_1a_2c_3$.

2. $a_3b_2c_1$
$c_1b_2a_3$.

3. $a_4b_3c_2d_1$
$d_1c_2b_3a_4$.

4. $a_1b_4c_3d_2$
$a_1d_2c_3b_4$.

5. $a_3b_2c_4d_1$
$d_1b_2a_3c_4$.

6. $a_3b_2c_1d_4$
$c_1b_2a_3d_4$.

7. Using the definition of a determinant, write the terms of the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Show by the use of the properties of determinants that each of the following determinants has the value zero:

8. $\begin{vmatrix} 5 & 3 \\ 0 & 0 \end{vmatrix}$.

9. $\begin{vmatrix} 0 & 7 \\ 0 & 4 \end{vmatrix}$.

10. $\begin{vmatrix} 3 & 3 \\ 4 & 4 \end{vmatrix}$.

11. $\begin{vmatrix} 3 & -1 & 3 \\ 2 & 4 & 2 \\ 5 & -3 & 5 \end{vmatrix}$.

12. $\begin{vmatrix} 1 & 6 & -6 \\ 5 & 2 & -2 \\ 4 & -3 & 3 \end{vmatrix}$.

13. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 2 & 1 \end{vmatrix}$.

14. $\begin{vmatrix} 1 & 2 & -5 \\ 2 & 4 & -4 \\ 0 & 0 & -3 \end{vmatrix}$.

15. $\begin{vmatrix} 0 & 1 & 0 \\ 5 & -6 & -7 \\ 0 & 2 & 0 \end{vmatrix}$.

16. $\begin{vmatrix} 7 & 6 & 5 \\ 6 & 5 & 4 \\ 5 & 4 & 3 \end{vmatrix}$.

Show that the following equations are true:

17. $\begin{vmatrix} 1 & -2 & 3 \\ 4 & -3 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 \\ -2 & -3 & 2 \\ 3 & 2 & 1 \end{vmatrix}$.

18. $\begin{vmatrix} 4 & 1 & 2 \\ 2 & 3 & 5 \\ 6 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 2 \\ 1 & 3 & 5 \\ 3 & 0 & 1 \end{vmatrix}$.

19. $\begin{vmatrix} 5 & -3 & 1 \\ 4 & 3 & 1 \\ 1 & -2 & 6 \end{vmatrix} = \begin{vmatrix} 9 & 0 & 2 \\ 4 & 3 & 1 \\ 1 & -2 & 6 \end{vmatrix}$.

20. $\begin{vmatrix} 2 & 6 & 4 \\ 3 & 6 & 9 \\ 1 & 1 & 1 \end{vmatrix} = 6\begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$.

21. $\begin{vmatrix} 2 & 4 & 5 \\ -3 & 7 & 1 \\ 5 & 6 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 4 & 5 \\ 5 & 7 & 1 \\ 4 & 6 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 4 & 5 \\ 2 & 7 & 1 \\ 9 & 6 & 0 \end{vmatrix}$.

22. $\begin{vmatrix} 16 & 14 \\ 18 & 21 \end{vmatrix} = \begin{vmatrix} 16 & 14 \\ 2 & 7 \end{vmatrix} = \begin{vmatrix} 14 & 7 \\ 2 & 7 \end{vmatrix} = \begin{vmatrix} 12 & 0 \\ 2 & 7 \end{vmatrix} = 84.$

145. Expansion by Minors. The definition of a determinant describes its expansion and hence provides a way of evaluating the determinant. But we shall now develop a more convenient method of evaluation.

The determinant which remains after the row and column of a given element are removed is called the **minor** of the element.

Illustration: In the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

the minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

For brevity, the minor of an element is customarily designated by the corresponding capital letter with the same subscript. For example, the minor of a_1 is A_1, the minor of b_2 is B_2, and the minor of c_2 is C_2.

We shall prove the following theorem:

THEOREM. *The expansion of a determinant may be obtained by taking the algebraic sum of the products of the elements of any row (or column) and their minors, when each product is given a plus or minus sign according as the sum of the numbers giving the row and column of the element is even or odd.*

Proof: We shall show first that the terms of a determinant D having a_1 as a factor are given by a_1A_1. Since each term of the expansion of D has one and only one factor from each row and each column, the terms of D involving a_1 have no other factor from the first row or the first column. Then the other factors come from the minor of a_1. Now if each term of the expansion of A_1 is multiplied by a_1, no change is made in the number of inversions of the subscripts of any product. Hence the terms of A_1 become terms of D when each is multiplied by a_1. Thus we see that the product of the element in the leading position (first row and first column) and its minor give the terms of D involving that element.

Next we wish to consider the product of an element and its minor if the element is in the ith row and jth column. This element can be transferred to the leading position in the following way: First, bring the ith row to row one by $i - 1$ interchanges of adjacent rows. Second, bring the jth column to column one by $j - 1$ interchanges of adjacent columns. It should be noticed that these interchanges in each case involve adjacent rows or adjacent columns, and hence do not disturb the minor of any element. Further, the interchanges of two rows or of two columns reverses the sign of the determinant. Then the $(i - 1) + (j - 1)$ interchanges produce $i + j - 2$ changes of the sign. This makes the original determinant D equal to the resulting determinant D' if $i + j - 2$ is even, and equal to its negative if $i + j - 2$ is odd. But $i + j - 2$ is even or odd according as

$i + j$ is even or odd. The element in the ith row and jth row column of D occupies the leading position in D'. Hence the product of the given element and its minor belongs to the expansion of D', according to the first part of the proof. But this product belongs also to D if $i + j$ is even, and its negative is a part of D if $i + j$ is odd. This is in agreement with the statement of the theorem.

Finally, we observe that the algebraic sum of the products of the elements of a row (or column) and their minors, with the sign in each case given according to the theorem, yields all the terms of the expansion involving the elements of the row (or column). From the definition of a determinant, these terms constitute the expansion, and hence the theorem is proved.

Expanding a determinant by minors expresses the determinant in terms of determinants of one less order. In turn, each of these could then be expressed as determinants of order one less, and so on. Thus a determinant of higher order may be reduced to second order determinants whose expansions could be easily written. The evaluation of determinants, however, is greatly facilitated by employing certain properties of determinants. A particularly helpful device is to obtain a determinant which has all of the elements, except one, of some row or column equal to zero. We shall illustrate with some examples.

Example 1. Find the value of the determinant

$$D = \begin{vmatrix} 4 & -6 & 5 \\ -3 & -6 & 5 \\ 7 & 3 & 1 \end{vmatrix}.$$

Solution. We first expand the determinant just as it is given. According to the elements of the first row the expansion is

$$D = 4\begin{vmatrix} -6 & 5 \\ 3 & 1 \end{vmatrix} + 6\begin{vmatrix} -3 & 5 \\ 7 & 1 \end{vmatrix} + 5\begin{vmatrix} -3 & -6 \\ 7 & 3 \end{vmatrix},$$

$$D = 4(-6 - 15) + 6(-3 - 35) + 5(-9 + 42),$$

$$D = 4(-21) + 6(-38) + 5(33),$$

$$D = -84 - 228 + 165 = -147.$$

Next let us obtain an equal determinant from the given determinant by subtracting the second row from the first row. Thus,

$$D = \begin{vmatrix} 4 & -6 & 5 \\ -3 & -6 & 5 \\ 7 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 0 & 0 \\ -3 & -6 & 5 \\ 7 & 3 & 1 \end{vmatrix}.$$

Now we expand the new determinant according to the elements of the first row, noticing that a zero element times its minor is zero.

$$D = 7\begin{vmatrix} -6 & 5 \\ 3 & 1 \end{vmatrix} + 0 + 0 = 7(-6 - 15) = 7(-21) = -147.$$

Example 2. Evaluate the determinant

$$D = \begin{vmatrix} 2 & -3 & 4 \\ 8 & 5 & 1 \\ -7 & 2 & -3 \end{vmatrix}.$$

Solution. Let us obtain an equivalent determinant which has all the elements, except one, of some row or column equal to zero. The element 1 in the second row, third column, makes this end easily achieved. We multiply the elements of the second row by 4 and subtract from the first row. Next we multiply the elements of the second row by 3 and add to the third row. This leaves the second row elements unchanged but gives new first and third rows. That is,

$$D = \begin{vmatrix} 2 & -3 & 4 \\ 8 & 5 & 1 \\ -7 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -30 & -23 & 0 \\ 8 & 5 & 1 \\ 17 & 17 & 0 \end{vmatrix}.$$

Taking advantage of the zero elements, we expand the new determinant according to the elements of the third column.

$$D = 0 - 1\begin{vmatrix} -30 & -23 \\ 17 & 17 \end{vmatrix} + 0 = -(-510 + 391) = 119.$$

Example 3. Evaluate the determinant

$$D = \begin{vmatrix} 2 & -12 & 7 & 5 \\ -4 & 6 & 4 & -3 \\ 10 & 0 & 1 & 2 \\ 6 & 9 & 5 & 0 \end{vmatrix}.$$

Solution. First factor 2 from the first column elements and 3 from the second column.

$$D = 6\begin{vmatrix} 1 & -4 & 7 & 5 \\ -2 & 2 & 4 & -3 \\ 5 & 0 & 1 & 2 \\ 3 & 3 & 5 & 0 \end{vmatrix}.$$

Three zeros can be obtained readily in the rows or columns which have the element 1. Let us get the zeros in the third row which already has one zero. We multiply the third column by 5 and subtract from the first column. Then we multiply the third column by 2 and subtract from the fourth column. This gives a determinant with new first and fourth columns.

$$D = 6\begin{vmatrix} -34 & -4 & 7 & -9 \\ -22 & 2 & 4 & -11 \\ 0 & 0 & 1 & 0 \\ -22 & 3 & 5 & -10 \end{vmatrix}$$

We expand according to the elements of the third row.

$$D = 6\begin{vmatrix} -34 & -4 & -9 \\ -22 & 2 & -11 \\ -22 & 3 & -10 \end{vmatrix}.$$

Next subtract the second row from the third row.

$$D = 6\begin{vmatrix} -34 & -4 & -9 \\ -22 & 2 & -11 \\ 0 & 1 & 1 \end{vmatrix}.$$

Now subtract the third column from the second column.

$$D = 6\begin{vmatrix} -34 & 5 & -9 \\ -22 & 13 & -11 \\ 0 & 0 & 1 \end{vmatrix}.$$

$$D = 6\begin{vmatrix} -34 & 5 \\ -22 & 13 \end{vmatrix} = 6(-442 + 110) = -1992.$$

Exercise 97

Evaluate the following determinants:

1. $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 4 & -5 \\ 4 & 0 & 1 \end{vmatrix}.$

2. $\begin{vmatrix} 6 & -3 & 7 \\ 7 & 5 & -9 \\ 6 & 3 & 7 \end{vmatrix}.$

3. $\begin{vmatrix} 5 & 0 & 1 \\ 3 & 4 & 2 \\ 6 & 1 & 3 \end{vmatrix}.$

4. $\begin{vmatrix} 0 & 5 & 4 & 1 \\ 3 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 4 \end{vmatrix}.$

5. $\begin{vmatrix} 6 & 8 & 7 & 2 \\ 3 & 2 & 4 & 1 \\ 2 & 3 & 8 & 2 \\ 5 & 4 & 2 & 1 \end{vmatrix}.$

6. $\begin{vmatrix} 3 & 2 & 2 & 3 \\ 3 & 1 & 1 & 1 \\ 6 & 2 & 2 & 1 \\ 9 & 0 & 1 & 1 \end{vmatrix}.$

7. $$\begin{vmatrix} 2 & -1 & 2 & 0 \\ 0 & -2 & 1 & 6 \\ 6 & 0 & 0 & 1 \\ 4 & 3 & 4 & 2 \end{vmatrix}.$$

8. $$\begin{vmatrix} 4 & 8 & 8 & 4 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 4 & 1 & 4 \end{vmatrix}.$$

9. $$\begin{vmatrix} 3 & 2 & 3 & 0 \\ 2 & 3 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{vmatrix}.$$

10. $$\begin{vmatrix} 3 & -9 & 0 & 12 \\ -2 & 1 & 3 & 0 \\ 3 & -2 & 1 & -3 \\ 0 & 2 & -3 & 1 \end{vmatrix}.$$

11. $$\begin{vmatrix} 6 & 6 & -1 & 1 \\ 3 & -1 & 1 & 1 \\ 3 & 1 & 6 & 1 \\ 6 & -1 & 1 & -6 \end{vmatrix}.$$

12. $$\begin{vmatrix} 1 & -2 & 1 & -1 & 3 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & -1 & -3 & 2 \\ 1 & 1 & 3 & 2 & 1 \\ 2 & -1 & 1 & 1 & 1 \end{vmatrix}.$$

13. $$\begin{vmatrix} 1 & 2 & 2 & 1 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 & 1 \\ 1 & 3 & 2 & 1 & 1 \\ 1 & 1 & 3 & 3 & 1 \end{vmatrix}.$$

146. Solution of Systems of Linear Equations by Determinants. In Section 51 systems of linear equations of two and three unknowns were solved by the use of determinants. The same method may be applied to systems of more unknowns. We shall show how determinants can be used to solve four linear equations of four unknowns, but it may be observed that the method of proof would also apply to the general case of n linear equations of n unknowns. Consider the system

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= e_1 \\ a_2x + b_2y + c_2z + d_2w &= e_2 \\ a_3x + b_3y + c_3z + d_3w &= e_3 \\ a_4x + b_4y + c_4z + d_4w &= e_4. \end{aligned}$$

We shall use D to stand for the determinant of the coefficients of the unknowns x, y, z, and w. And N_x, N_y, N_z, and N_w will stand for the four determinants obtained from D by replacing the coefficients of x, y, z, and w, in turn, by the corresponding constant terms. Then the solution of the given system of equations, as we shall show, may be written as

$$x = \frac{N_x}{D}, \quad y = \frac{N_y}{D}, \quad z = \frac{N_z}{D}, \quad w = \frac{N_w}{D},$$

if $D \neq 0$. If we multiply both members of

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

by x, we get

$$Dx = \begin{vmatrix} a_1x & b_1 & c_1 & d_1 \\ b_1x & b_2 & c_2 & d_2 \\ c_1x & b_3 & c_3 & d_3 \\ d_1x & b_4 & c_4 & d_4 \end{vmatrix}.$$

Now we multiply the second column of the determinant by y, the third column by z, and the fourth column by w and add each of the products to the first column to obtain

$$Dx = \begin{vmatrix} a_1x + b_1y + c_1z + d_1w & b_1 & c_1 & d_1 \\ a_2x + b_2y + c_2z + d_2w & b_2 & c_2 & d_2 \\ a_3x + b_3y + c_3z + d_3w & b_3 & c_3 & d_3 \\ a_4x + b_4y + c_4z + d_4w & b_4 & c_4 & d_4 \end{vmatrix}.$$

The elements of the first column of this determinant are the left members of the given system of equations, and hence may be replaced by their equals as found in the right members. We have then

$$Dx = \begin{vmatrix} e_1 & b_1 & c_1 & d_1 \\ e_2 & b_2 & c_2 & d_2 \\ e_3 & b_3 & c_3 & d_3 \\ e_4 & b_4 & c_4 & d_4 \end{vmatrix} = N_x,$$

and $x = N_x/D$.

Similarly, we may find

$$y = \frac{N_y}{D}, \quad z = \frac{N_z}{D}, \quad w = \frac{N_w}{D}.$$

Then we have a solution of the given system of equations if these values for the unknowns x, y, z, and w satisfy each of the equations. Suppose that we check the first equation. The remaining equations could be checked in the same way. Substituting, we have

$$\frac{a_1N_x}{D} + \frac{b_1N_y}{D} + \frac{c_1N_z}{D} + \frac{d_1N_w}{D} = e_1.$$

Now transposing the constant term to the left side and multiplying by D gives

$$a_1N_x + b_1N_y + c_1N_z + d_1N_w - e_1D = 0.$$

If this expression on the left is equal to zero, the first equation of the given system is satisfied. The expression is the expansion of a determinant.

That is,

$$a_1N_x + b_1N_y + c_1N_z + d_1N_w - e_1D = \begin{vmatrix} a_1 & e_1 & b_1 & c_1 & d_1 \\ a_1 & e_1 & b_1 & c_1 & d_1 \\ a_2 & e_2 & b_2 & c_2 & d_2 \\ a_3 & e_3 & b_3 & c_3 & d_3 \\ a_4 & e_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

as may be verified by expanding the determinant according to the elements of the first row. But the determinant is equal to zero since the first and second rows are identical.

Example. Solve the system

$$\begin{aligned} 2x \phantom{{}+2y} + z - w &= 1 \\ x + 2y - 2z + 3w &= 9 \\ 3x - 2y + 3z - 2w &= 2 \\ 4x - 4y - 2z - 3w &= 3. \end{aligned}$$

Solution. The solution may be had from the five determinants D, N_x, N_y, N_z, and N_w. We show how D may be evaluated.

$$D = \begin{vmatrix} 2 & 0 & 1 & -1 \\ 1 & 2 & -2 & 3 \\ 3 & -2 & 3 & -2 \\ 4 & -4 & -2 & -3 \end{vmatrix}$$

We add twice the fourth column to the first column, and add the fourth column to the third column, and then expand by the elements of the first row; whence,

$$D = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 7 & 2 & 1 & 3 \\ -1 & -2 & 1 & -2 \\ -2 & -4 & -5 & -3 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 1 \\ -1 & -2 & 1 \\ -2 & -4 & -5 \end{vmatrix} = 2\begin{vmatrix} 7 & 1 & 1 \\ -1 & -1 & 1 \\ -2 & -2 & -5 \end{vmatrix}.$$

Referring to the determinant on the right, we add the third column to the first and also to the second column and obtain

$$D = 2\begin{vmatrix} 8 & 2 & 1 \\ 0 & 0 & 1 \\ -7 & -7 & -5 \end{vmatrix} = -2\begin{vmatrix} 8 & 2 \\ -7 & -7 \end{vmatrix} = 84.$$

By evaluating it may be verified that

$$N_x = \begin{vmatrix} 1 & 0 & 1 & -1 \\ 9 & 2 & -2 & 3 \\ 2 & -2 & 3 & -2 \\ 3 & -4 & -2 & -3 \end{vmatrix} = 168; \quad N_y = \begin{vmatrix} 2 & 1 & 1 & -1 \\ 1 & 9 & -2 & 3 \\ 3 & 2 & 3 & -2 \\ 4 & 3 & -2 & -3 \end{vmatrix} = -84;$$

$$N_z = \begin{vmatrix} 2 & 0 & 1 & -1 \\ 1 & 2 & 9 & 3 \\ 3 & -2 & 2 & -2 \\ 4 & -4 & 3 & -3 \end{vmatrix} = 0; \qquad N_w = \begin{vmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & -2 & 9 \\ 3 & -2 & 3 & 2 \\ 4 & -4 & -2 & 3 \end{vmatrix} = 252.$$

Hence,

$$x = \tfrac{168}{84} = 2, \quad y = \frac{-84}{84} = -1, \quad z = \tfrac{0}{84} = 0, \quad w = \tfrac{252}{84} = 3.$$

Exercise 98

Solve by determinants and check your results:

1. $$\begin{aligned} 3x - 5y &= 7 \\ x + y &= 2. \end{aligned}$$

2. $$\begin{aligned} 4x - 3y &= 1 \\ 3x - 2y &= 7. \end{aligned}$$

3. $$\begin{aligned} 2x + 3y + z &= 9 \\ 4x + y &= 7 \\ x - 3y - 7z &= 6. \end{aligned}$$

4. $$\begin{aligned} 3x + y + z &= 2 \\ x + y + z &= 0 \\ 5x - 2y + 3z &= -8. \end{aligned}$$

5. $$\begin{aligned} 3x - 5z &= -1 \\ 2x + 7y &= 6 \\ x + y + z &= 5. \end{aligned}$$

6. $$\begin{aligned} x + 5y - z &= 9 \\ 3x - 3y + 2z &= 7 \\ 2x - 4y + 3z &= 1. \end{aligned}$$

7. $$\begin{aligned} -x + 2y + 3z - 2w &= 2 \\ 3x + y + z - w &= 4 \\ 2x - y - z + 2w &= 0 \\ x + y + z + w &= 0. \end{aligned}$$

8. $$\begin{aligned} 2x + 2y - z - w &= 1 \\ 2x - 4y + 3z + 2w &= 8 \\ 3x + 2y - 2z - 3w &= 7 \\ x + y - z + w &= 0. \end{aligned}$$

9. $$\begin{aligned} 2x - 3z + w &= 1 \\ x - y + 2w &= 1 \\ -3y + z + w &= 1 \\ x + y + z &= 1. \end{aligned}$$

10. $$\begin{aligned} x - y + z - w &= 3 \\ x + y + z + w &= -5 \\ x - 3y - z - w &= 9 \\ x + y - z + w &= 1. \end{aligned}$$

11. $$\begin{aligned} x + y + 2z + 2w &= 3 \\ 2x - y - z + 2w &= 1 \\ 2x + 2y - 3z + w &= 2 \\ 4x - 2y + z + 3w &= 6. \end{aligned}$$

12. $$\begin{aligned} x + y + 2z + 3w &= -2 \\ 2x + y - 3z - 2w &= -1 \\ 3x + 2y - z - w &= -5 \\ 2x + 3y - z - 2w &= 1. \end{aligned}$$

13. $$\begin{aligned} x + y + z + w &= 2 \\ x - 2y + 2z + 2w &= -6 \\ 2x + y - 2z + 2w &= -5 \\ 3x - y + 3z - 3w &= -3. \end{aligned}$$

14. $$\begin{aligned} 2x + 3y + z + w &= 2 \\ 2x - 2y + z - w &= 7 \\ x - y - 3z - w &= 2 \\ 3x + 2y + 2z + 2w &= 5. \end{aligned}$$

147. Linear Systems with More Equations Than Unknowns. The systems of linear equations which we have considered thus far have had the same number of equations as unknowns. Systems of equations sometimes arise, however, which do not have this balance between the equations and unknowns. A detailed discussion of this problem is beyond our present stage. Accordingly we shall restrict ourselves to the special case in which the number of equations is one more than the number of unknowns.

To illustrate this case we choose the system

$$a_1x + b_1y = c_1 \tag{1}$$

$$a_2x + b_2y = c_2 \tag{2}$$

$$a_3x + b_3y = c_3. \tag{3}$$

Here we have three equations and two unknowns. Usually systems of this kind are inconsistent (that is, do not have a solution). However, if any two of the equations are consistent and their solution satisfies the other equation, then the system of three equations is consistent.

Let us assume that equations (1) and (2) are neither dependent nor inconsistent. This means that

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

The solution of the two equations is

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}.$$

We now substitute these values for x and y in equation (3). This gives

$$\frac{a_3\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} + \frac{b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D} = a_3.$$

By clearing of fractions, changing signs, and interchanging columns in one of the determinants, we reduce the equation to

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The left side of this equation is the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Hence this determinant must be equal to zero if the given system of equations is consistent. Conversely, if this determinant is equal to zero and any two of the given equations are consistent, then the three equations form a consistent system.

Systems with one extra equation in more than two unknowns could be treated in an analogous way. In this connection we state a general theorem.

THEOREM. *A system of* $(n + 1)$ *linear equations in* n *unknowns has a single solution if and only if the following conditions hold:*

1. The determinant whose $(n + 1)$ *columns are formed from the coefficients of the* n *unknowns and the constant terms is equal to zero.*

2. The determinant of the coefficients of some n *of the equations is different from zero.*

Example. Determine if the following system of equations is consistent.

$$\begin{aligned} x + y + z &= 0 \\ x + y + 3z &= 2 \\ 2x - 3y - 5z &= 8 \\ 3x - 2y - 8z &= 4. \end{aligned}$$

Solution. We first evaluate the determinant formed by the coefficients of the unknowns and the constant terms. Thus we find

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 \\ 2 & -3 & -5 & 8 \\ 3 & -2 & -8 & 4 \end{vmatrix} = 0.$$

This shows that one part of the theorem is satisfied. We next determine if the second part of the theorem is satisfied. Trying the first three of the given equations, we find

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & -3 & -5 \end{vmatrix} = 10.$$

Since this determinant is different from zero, the three equations have a solution. The solution, as may be verified, is $x = 2$, $y = -3$, $z = 1$. These values also satisfy the fourth equation.

148. Systems of Homogeneous Linear Equations. A linear equation is said to be **homogeneous** if the constant term is zero. The equations

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \\ a_3x + b_3y + c_3z &= 0 \end{aligned}$$

constitute a system of homogeneous linear equations. Obviously the equations are satisfied by substituting zero for each unknown. This solution may also be obtained by using determinants if the determinant of the coefficients of the unknown D is not equal to zero. We may write the solution as

$$x = \frac{N_x}{D}, \quad y = \frac{N_y}{D}, \quad z = \frac{N_z}{D}.$$

N_x, N_y, and N_z stand for determinants as defined in Section 146. Each of these numerator determinants, having a column of zeros, is equal to zero. Hence, with $D \neq 0$, this is the only possible solution.

We see at once that any system of homogeneous linear equations is satisfied by the value zero for each unknown. This is called a **trivial solution.** We raise the question as to the existence of solutions other than the trivial solution. Relative to this question, we state a theorem.

THEOREM. *A system of n homogeneous linear equations in n unknowns has nontrivial solutions if and only if the determinant of the coefficients is equal to zero.*

This theorem is proved in textbooks on theory of equations. Although we do not include a proof, we shall illustrate a method for finding nontrivial solutions.

Example. Find nontrivial solutions of the system

$$\begin{aligned} x - y - 3z &= 0 \\ 2x - 2y - 6z &= 0 \\ 2x + 3y - z &= 0. \end{aligned}$$

Solution. We first evaluate D. Thus we find

$$D = \begin{vmatrix} 1 & -1 & -3 \\ 2 & -2 & -6 \\ 2 & 3 & -1 \end{vmatrix} = 0.$$

Since $D = 0$, there are nontrivial solutions. We next find the solutions.

Transposing the z terms in the second and third equations gives

$$\begin{aligned} 2x - 2y &= 6z \\ 2x + 3y &= z. \end{aligned}$$

The solution of these equations for x and y in terms of z is $x = 2z$, $y = -z$. These values for x and y satisfy all three of the given equations. Hence an indefinite number of solutions can be found simply by assigning z any value and using the values $2z$ and $-z$ for x and y. Giving z the literal number k, we write the solution as

$$x = 2k, \quad y = -k, \quad z = k,$$

where k can be any number.

Exercise 99

Test each system of equations for consistency or inconsistency. Find the solution of each system whose equations are consistent.

1. $\begin{aligned} 4x - 3y &= 3 \\ 2x - 3y &= 2 \\ 2x - 6y &= 3. \end{aligned}$

2. $\begin{aligned} 4x - 2y &= 3 \\ 2x - 3y &= 2 \\ 6x - 5y &= 4. \end{aligned}$

3. $\begin{aligned} 3x - 3y &= 4 \\ 2x + 2y &= 3 \\ 7x + y &= 10. \end{aligned}$

4. $\begin{aligned} x + 7y &= 10 \\ 2x + 5y &= 2 \\ x + 16y &= 28. \end{aligned}$

5. $\begin{aligned} 2x + 3y &= -1 \\ 3x - 4y &= 24 \\ x + 10y &= -26. \end{aligned}$

6. $\begin{aligned} 6x - 5y &= 14 \\ 4x + 3y &= -13 \\ 2x + y &= 0. \end{aligned}$

7. $\begin{aligned} 2x - y + 2z &= 6 \\ 3x + y + z &= 8 \\ x - 2y + z &= 0 \\ x + y &= 1. \end{aligned}$

8. $\begin{aligned} 2x + y - z &= 8 \\ 3x - 4y - 5z &= 8 \\ 5x + y + z &= 10 \\ x + y + z &= 2. \end{aligned}$

9. $\begin{aligned} x - y &= 1 \\ x + z &= -6 \\ y - z &= 1 \\ x + y - 2z &= 3. \end{aligned}$

10. $\begin{aligned} 2x + 3y + z &= 9 \\ x - 2y - 3z &= 1 \\ 5x + 4y + 6z &= 5 \\ x + y - z &= 6. \end{aligned}$

11. $\begin{aligned} x - 2y - z &= 3 \\ x + 3y + z &= 4 \\ x - y - 2z &= 5 \\ x + y + 2z &= 6. \end{aligned}$

12. $\begin{aligned} 2x - y + z &= 0 \\ 2x + y - z &= 2 \\ 2x - y - z &= 3 \\ 2x + y + z &= 4. \end{aligned}$

Test each system of equations for the existence of nontrivial solutions. Solve the systems which have nontrivial solutions.

13. $\begin{aligned} x + 2y &= 0 \\ 2x - y &= 0. \end{aligned}$

14. $\begin{aligned} 3x - 2y &= 0. \\ 6x - 4y &= 0. \end{aligned}$

15. $\begin{aligned} 3x + 3y &= 0 \\ 2x + 2y &= 0. \end{aligned}$

16. $\begin{aligned} x + y + z &= 0 \\ x - y - 5z &= 0 \\ x + 2y + 4z &= 0. \end{aligned}$

17. $\begin{aligned} 2x - 3y - z &= 0 \\ x + 3y - 2z &= 0 \\ x - 3y &= 0. \end{aligned}$

18. $\begin{aligned} x + 2y - 5z &= 0 \\ x - y + 4z &= 0 \\ x + 4y - 9z &= 0. \end{aligned}$

19. $\begin{aligned} x + y + z + w &= 0 \\ x - 7y + z - w &= 0 \\ x + y - 7z - w &= 0 \\ x - y + 3z + w &= 0. \end{aligned}$

20. $\begin{aligned} 4x - y - 2z + w &= 0 \\ x - y - z + w &= 0 \\ 5x + y - z - w &= 0 \\ 2x + 2y + 2z - 3w &= 0. \end{aligned}$

Table I.—Powers and Roots

No.	Sq.	Sq. Root	Cube	Cube Root	No.	Sq.	Sq. Root	Cube	Cube Root
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	**52**	2,704	7.211	140,608	3.733
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	**55**	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	**57**	3,249	7.550	185,193	3.849
8	64	2.828	512	2.000	**58**	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	**65**	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	**68**	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	**69**	4 761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	**70**	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	**71**	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	**73**	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	**77**	5,929	8.775	456,533	4.254
28	784	5.292	21,952	3.037	**78**	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	**81**	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	**82**	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	**83**	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	**84**	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	**85**	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	**86**	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	**88**	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	**89**	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	**90**	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	**92**	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.644	804,357	4.531
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	**95**	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	**96**	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	**97**	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.642

TABLE II.—COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

Table II.—Common Logarithms (*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

TABLE III.—AMERICAN EXPERIENCE MORTALITY TABLE

Age	Number living	Number dying	Yearly probability of dying	Yearly probability of living	Age	Number living	Number dying	Yearly probability of dying	Yearly probability of living
10	100 000	749	0.007 490	0.992 510	53	66 797	1 091	0.016 333	0.983 667
11	99 251	746	0.007 516	0.992 484	54	65.706	1 143	0.017 396	0.982 604
12	98 505	743	0.007 543	0.992 457	55	64 563	1 199	0.018 571	0.981 429
13	97 762	740	0.007 569	0.992 431	56	63 364	1 260	0.019 885	0.980 115
14	97 022	737	0.007 596	0.992 404	57	62 104	1 325	0.021 335	0.978 665
15	96 285	735	0.007 634	0.992 366	58	60.779	1 394	0.022 936	0.977 064
16	95 550	732	0.007 661	0.992 339	59	59 385	1 468	0.024 720	0.975 280
17	94 818	729	0.007 688	0.992 312	60	57 917	1 546	0.026 693	0.973 307
18	94 089	727	0.007 727	0.992 273	61	56 371	1 628	0.028 880	0.971 120
19	93 362	725	0.007 765	0.992 235	62	54 743	1 713	0.031 292	0.968 708
20	92 637	723	0.007 805	0.992 195	63	53 030	1 800	0.033 943	0.966 057
21	91 914	722	0.007 855	0.992 145	64	51 230	1 889	0.036 873	0.963 127
22	91 192	721	0.007 906	0.992 094	65	49 341	1 980	0.040 129	0.959 871
23	90 471	720	0.007 958	0.992 042	66	47 361	2 070	0.043 707	0.956 293
24	89 751	719	0.008 011	0.991 989	67	45 291	2 158	0.047 647	0.952 353
25	89 032	718	0.008 065	0.991 935	68	43 133	2 243	0.052 002	0.947 998
26	88 314	718	0.008 130	0.991 870	69	40 890	2 321	0.056 762	0.943 238
27	87 596	718	0.008 197	0.991 803	70	38 569	2 391	0.061 993	0.938 007
28	86 878	718	0.008 264	0.991 736	71	36 178	2 448	0.067 665	0.932 335
29	86 160	719	0.008 345	0.991 655	72	33 730	2 487	0.073 733	0.926 267
30	85 441	720	0.008 427	0.991 573	73	31 243	2 505	0.080 178	0.919 822
31	84 721	721	0.008 510	0.991 490	74	28 738	2 501	0.087 028	0.912 972
32	84 000	723	0.008 607	0.991 393	75	26 237	2 476	0.094 371	0.905 629
33	83 277	726	0.008 718	0.991 282	76	23 761	2 431	0.102 311	0.897 689
34	82 551	729	0.008 831	0.991 169	77	21 330	2 369	0.111 064	0.888 936
35	81 822	732	0.008 946	0.991 054	78	18 961	2 291	0.120 827	0.879 173
36	81 090	737	0.009 089	0.990 911	79	16 670	2 196	0.131 734	0.868 266
37	80 353	742	0.009 234	0.990 766	80	14 474	2 091	0.144 466	0.855 534
38	79 611	749	0.009 408	0.990 592	81	12 383	1 964	0.158 605	0.841 395
39	78 862	756	0.009 586	0.990 414	82	10 419	1 816	0.174 297	0.825 703
40	78 106	765	0.009 794	0.990 206	83	8 603	1 648	0.191 561	0.808 439
41	77 341	774	0.010 008	0.989 992	84	6 955	1 470	0.211 359	0.788 641
42	76 567	785	0.010 252	0.989 748	85	5 485	1 292	0.235 552	0.764 448
43	75 782	797	0.010 517	0.989 483	86	4 193	1 114	0.265 681	0.734 319
44	74 985	812	0.010 829	0.989 171	87	3 079	933	0.303 020	0.696 980
45	74 173	828	0.011 163	0.988 837	88	2 146	744	0.346 692	0.653 308
46	73 345	848	0.011 562	0.988 438	89	1 402	555	0.395 863	0.604 137
47	72 497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71 627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 532
49	70 731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69 804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68 842	1 001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67 841	1 044	0,015 389	0.984 611	95	3	3	1,000 000	0.000 000

Table IV.—Trigonometric Functions

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.018	.999	.018	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.998	.052	48°	.743	.669	1.111
4°	.070	.997	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.994	.105	51°	.777	.629	1.235
7°	.122	.992	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.144
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.992	.122	8.144
39°	.629	.777	.810	84°	.994	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.997	.070	14.300
42°	.669	.743	.900	87°	.998	.052	19.081
43°	.682	.731	.933	88°	.999	.035	28.636
44°	.695	.719	.966	89°	.999	.018	57.290
				90°	1.000	.000	——

ANSWERS

Answers are not included to the problems whose numbers are multiples of three.

Exercise 1

1. $20.$

2. $8.$

4. $-30.$

5. $-1.$

7. $-8.$

8. $8.$

10. $-6a.$

11. $0.$

13. $4xy.$

14. $-5x.$

16. $-9.$

17. $-10.$

19. $6.$

20. $0.$

22. $-8a.$

23. $3x.$

25. $11mn.$

26. $6mn.$

28. $8.$

29. $23.$

31. $-11.$

32. $-3.$

34. $12.$

35. $0.$

37. $x.$

38. $8x.$

40. $5mn.$

41. $-10mn.$

43. $20.$

44. $-10.$

46. $0.$

47. $-6x.$

49. $2c.$

50. $-5c.$

52. $-8a.$

53. $4x - 2y + 1;\ -2x + 6y - 17.$

55. $3a + 2b - 4c;\ a - 8b + 2c.$

56. $3a - b - 4c;\ 5a - 11b - 10c.$

58. $5x + 2y - z;\ x - 8y - z.$

59. $10x - 2y + 6z;\ 4x + 16y - 6z.$

61. $14m - 15n - 8;\ 8m + n + 34.$

62. $23a - 3b + 8c;\ -3a - 31b + 40c.$

64. $-5a + 2b + 32c;\ -29a + 28b.$

65. $33x - 22y - 4z;\ -x - 24z.$

67. $8x + 13xy - 3yz;\ 2x - xy + 9yz.$

68. $3a + 13b - c - 15;\ -7a + b + 7c + 7.$

70. $11x - 4y - 1;\ 3x + 6y - 15.$

71. $6a + 8b + 8c;\ 8a - 6c.$

73. $22x - 12y - 11;\ 2x + 10y + 15.$

74. $-17x - 39;\ -5x - 10y + 21.$

76. $12x - 15y + 11z;\ -18x + 23y - 17z.$

77. $a + 15b + 34c;\ -15a - b + 8c.$

79. $7a - 4b + 15c.$

80. $19x + 5y + 16z;\ -9x - 9y - 16z.$

82. $9x - y - 10z + 4.$

83. $8a + 12b - 7c - 5.$

85. $3xy + 6x + 7y - 4.$

86. $-5a + 10c - 2y + 1.$

88. $-5ab + 9ac - 2bc - 5.$

89. $7ac + 5a + 3b - 4.$

91. $-xy - 1.$

92. $10m + 5a + 3c - 6.$

Exercise 2

1. $4a - 2c.$

2. $2x + y - 4.$

4. $y.$

5. $2a - b + 6.$

7. $10a - 4b.$

8. $3x + 1.$

10. $-10a + 6b - 2c.$

11. $6xy + 6x - 3y.$

13. $19x + 9y + 2.$

14. $15x - 13y.$

Exercise 3

1. 40.
2. -21.
4. $6a^2b$.
5. $-28ab$.
7. $6a^3$.
8. $-12m^3n^2$.
10. $-6x^2y^2$.
11. $24a^2b^2$.
13. $-a^3b^4$.
14. $10x^2y^2$.
16. $4x^2 - 12xy$.
17. $-6a^2 + 4ab$.
19. $-21a^3 + 28a^2$.
20. $x^3y - xy^3$.
22. $x^2 - 9y^2$.
23. $4a^2 - b^2$.
25. $x^2 - x - 2$.
26. $2a^2 + 5ab - 3b^2$.
28. $2m^2 - 11mn + 12n^2$.
29. $12m^2 - 9m - 30$.
31. $12x^4 - 21x^3 + 6x^2$.
32. $2a^3b - 8a^2b^2 + 6a^2b^3$.
34. $-6x^2y^2 - 9x^2y^3 + 12xy^3 - 3xy^2$.
35. $-12mn + 20m^2n - 24m^2n^2 - 12mn^2$.
37. $14xy^2 - 28x^2y^2 + 7xy^3 + 21xy$.
38. $-x^5y^2 + 2x^3y^3 - 3x^3y^4$.
40. $a^3 + b^3$.
41. $8a^3 + 27$.
43. $y^3 + 3xy^2 + x^2y - 2x^3$.
44. $2c^3 - 13ac^2 + 21a^2c - 4a^3$.
46. $3a^3 + 23a^2c - 10ac^2 - 16c^3$.
47. $2b^4 - 3b^3 + 9b^2 - 6b + 10$.
49. $a^2 + 2a + 1 - b^2$.
50. $4x^2 + 12x + 9 - y^2$.
52. $x^4 + x^3 - 5x^2 + 13x - 6$.
53. $a^4 + a^3 - 5a^2 - 17a - 12$.
55. $6x^4 - x^3 - 14x^2 - 11x - 28$.
56. $3m^4 - 11m^3 - 17m^2 + 13m + 12$.
58. $a^3 + 4a^2 + a - 6$.
59. $2x^3 + 5x^2 - 28x - 15$.
61. $6 - 17m + 14m^2 - 3m^3$.
62. $64 - 32x - 4x^2 + 2x^3$.

Exercise 4

1. 1.
2. -1.
4. b^2.
5. a^3.
7. $-y$.
8. z.
10. $2a^3$.
11. $-3d$.
13. $2ab$.
14. $-5xy$.
16. $2x - 1$.
17. $1 - 3x$.
19. $3x^2 - 2x - 1$.
20. $m - n + 1$.
22. $5x^2 - x - 2$.
23. $-2a^2 + a + 2$.
25. $x - 4$.
26. $x - 9 + \dfrac{2}{x+2}$.
28. $3x + 4$.
29. $4x + 3 + \dfrac{2}{3x-4}$.
31. $x^2 - 3x + 3 - \dfrac{1}{x+3}$.
32. $x^2 + 2x + 8$.
34. $a^2 + ab + b^2$.
35. $a^2 - ab + b^2$.
37. $x + 9$.
38. $6x + 2 + \dfrac{-15x+2}{2x^2+x+1}$.
40. $2x^3 + 2x^2 - x + \dfrac{1}{2x-3}$.
41. $2x^2 + x - 3$.
43. $4m^2 - 2mn + n^2$.
44. $m^2 - n^2$.
46. $x^4 - 3x^3 + x^2 + 3x - 2 + \dfrac{3}{x-2}$.
47. $x^3 - x + 4 + \dfrac{8x-2}{x^2+x+2}$.
49. $x^3 - x^2 + x + 4$.

Exercise 5

1. $m^2 - 16$.
2. $4x^2 - 9$.
4. $25 - m^2$.
5. $4m^2 - n^2$.
7. $16 - 9x^2$.
8. $x^4 - y^4$.
10. $a^2b^2 - c^2d^2$.
11. $4x^2 - y^4$.
13. 2496.
14. 6375.
16. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.
17. $\frac{9}{16}x^2 - \frac{4}{25}y^2$.
19. $9a^4 - 121b^4$.
20. $16a^4 - 49c^4$.
22. $9a^6 - 16b^8$.
23. $\dfrac{1}{9x^2} - \dfrac{49}{25y^2}$.
25. $\dfrac{36x^4}{25y^4} - \dfrac{25y^4}{36x^4}$.
26. $\dfrac{4a^6}{9b^4} - \dfrac{9x^4}{4y^6}$.

Exercise 6

1. $x^2 + 2x + 1$.
2. $x^2 - 4x + 4$.
4. $4m^2 - 4m + 1$.
5. $m^2 + 6mn + 9n^2$.
7. $4x^2 + 12xy + 9y^2$.
8. $x^2 + 4xy + 4y^2$.
10. $49m^2 - 70mn + 25n^2$.
11. $x^2y^2 - 6xyz + 9z^2$.
13. 3721.
14. 2401.
16. $4a^4 + 4a^2b^2 + b^4$.
17. $a^6 - 2a^3b^3 + b^6$.

19. $m^4n^2 - 8m^2n + 16$.
20. $81 + 36n^2 + 4n^4$.
22. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$.
23. $x^2 + 4y^2 + z^2 + 4xy + 2xz + 4yz$.
25. $a^2 + 4b^2 + 9 - 4ab - 6a + 12b$.
26. $16 + m^2 + 9n^2 - 8m - 24n + 6mn$.

28. $a^4 + b^4 + 4 + 2a^2b^2 + 4a^2 + 4b^2$.
29. $x^2 + y^2 + z^2 + w^2 + 2xy + 2xz - 2xw + 2yz - 2yw - 2zw$.
31. $4a^2 + b^2 + 9c^2 + d^2 - 4ab + 12ac - 4ad - 6bc + 2bd - 6cd$.
32. $9m^2 + n^2 + p^2 + 4q^2 - 6mn - 6mp + 12mq + 2np - 4nq - 4pq$.

34. $a^2 + 2ab + b^2 - 1$.
35. $x^2 - 2xy + y^2 - 4$.
37. $9m^2 + 12mn + 4n^2 - 1$.
38. $1 - 6m + 9m^2 - n^2$.
40. $x^2 - 25y^2 + 10y - 1$.
41. $x^4 + x^2 + 1$.

Exercise 7

1. $x^2 + 3x + 2$.
2. $x^2 - x - 20$.
4. $x^2 - 8x + 15$.
5. $x^2 - 3xy + 2y^2$.
7. $20 + 31m + 12m^2$.
8. $9m^2 - 3mn - 2n^2$.
10. $10a^2 - 17ab + 3b^2$.
11. $15a^2 - 17ab - 4b^2$.
13. $-7a^2 - 43ab - 6b^2$.
14. $-5a^2 - 24ab + 5b^2$.
16. $2x^4 - 9x^2 + 4$.
17. $12x^4 - 23x^2 - 24$.
19. $x^3 + 2x^2 + 3x + 2$.
20. $x^3 + x^2 - 5x - 6$.
22. $6m^3 + 10m^2 - 12m - 20$.

23. $8m^3 - 22m^2 + 15m$.
25. $x^2 + 2xy + y^2 + 4x + 4y + 3$.
26. $x^2 - 2xy + y^2 - x + y - 6$.
28. $6x^2 + 12xy + 6y^2 - x - y - 1$.

Exercise 8

1. $x^3 - 3x^2 + 3x - 1$.
2. $x^3 - 9x^2 + 27x - 27$.
4. $27 + 27x + 9x^2 + x^3$.

5. $8x^3 + 12x^2 + 6x + 1$.
7. $125 - 75x + 15x^2 - x^3$.
8. $1 - 15x + 75x^2 - 125x^3$.
10. $8a^3 - 36a^2b + 54ab^2 - 27b^3$.
11. $-a^3 - 3a^2b - 3ab^2 - b^3$.
13. $125 + 75ab + 15a^2b^2 + a^3b^3$.
14. $a^6 - 9a^4b + 127a^2b^2 - 26b^3$.
16. $216m^3n^3 - 108m^2n^2y + 18mny^2 - y^3$.
17. $27m^3 + 108m^2xy + 144mx^2y^2 + 64x^3y^3$.

Supplementary Problems

1. $7a - b$; $-a - 7b$.
2. $12x - 10y$; $2x + 6y$.
4. $10c + 9d$; $-4c - 17d$.

5. $6x + 2y - 6$; $-2x + 4y - 8$.
7. $3m - 5n + 11$; $m + 7n - 1$.

8. $4x - 2y$; $-2x - 4y - 2z$.
10. $5a - 4b + 5c - 3$; $-3a - 2b - 19c + 15$.
11. $-4p + r - 20$; $-10p - 8q + 13r + 6$.

13. $14x - 10y + 5$; $-14y + 5$.
14. $a + 12b + 3c$; $-9a + 14b + 11c$.
16. $-a + 6b - 9c$; $-5a + 10b + 13c$.
17. $9p - 5q - 9r$; $p - 13q - 19r$.

19. $9a - 2b - 2c - 7;\ -a + 8b + 10c + 7.$
20. $-3p - q - 4r - 8;\ -5p + 3q + 2r.$

22. $-2a - 2b.$
23. $4a + 2b.$
25. $8p - 5q.$
26. $-2r.$
28. $-2 + 2b.$
29. $x - 6y.$
31. $2m + 4n.$
32. $9x - 2y.$
34. $13x + 3y - 13z + 10.$
35. $4c + y - 8.$
37. $2x + 5y - 2.$
38. $a - 6b + 7c.$
40. $-2p + 2x - y + 2.$
41. $6x + 9y - 2a + 10.$
43. $12m^3 - 8m^2.$
44. $6a^2b - 4ab^2.$
46. $9x^2 - y^2.$
47. $16a^2 - 25.$
49. $4x^4 - y^4.$
50. $9m^2n^2 - 16.$
52. $x^2 + 4x + 3.$
53. $a^2 - 3a + 2.$
55. $2m^2 + 3mn - 2n^2.$
56. $4m^2 - mn - 3n^2.$
58. $2 + 3x - 20x^2.$
59. $4m^2 - 12mn + 9n^2.$
61. $a^3 + 2a^2 + 2a + 1.$

62. $2a^3 - 3a^2 + 1.$
64. $2x^3 - 6x^2 + 5x - 1.$
65. $6m^3 - 7m^2 - 11m + 12.$
67. $14x^2 - 7xy.$
68. $12x^2 + 20xy - 6y^2.$
70. $-a^2b + 2ab.$
71. $12x^4 - 3x^3 - 23x^2 + 2x + 10.$
73. $a^2 + 4ab + 4b^2 - 1.$
74. $4a^2 - 9b^2 - 12b - 4.$
76. $24m^4 + 6m^3 + 23n^2 + 7m + 6.$
77. $6x^4 - x^3 - x^2 + 43x - 35.$
79. $2y^4 - 11y^3 - 27y^2 - y + 12.$

80. $3x^3 - 10x^2y + 5xy^2 + 4y^3 + 2x^2 - 4xy - 2y^2.$
82. $2m^4 + m^3n - 3m^2n^2 + 5mn^3 + 7mn - 2n^4 - 2m^2 - 6n^2 - 4.$

83. $3x + y.$
85. $2x - 3y.$
86. $b + 3c.$
88. $-x^2 - 2x + 3.$
89. $x^2 - xy + y^2.$
91. $2m^2 + 3m - 2.$

92. $x^2 - x + 1.$
94. $x^4 - x^3 + x^2 + 3x - 3.$

95. $x^2 + 4xy + 4y^2.$
97. $4x^2 - 20xy + 25y^2.$
98. $16 + 16y + 4y^2.$
100. $x^4 - 4x^2y + 4y^2.$
101. $9m^2 - 6mn^2 + n^4.$
103. $4x^4 - 4x^2y^3 + y^6.$
104. $27 - 27x + 9x^2 - x^3.$
106. $1 - 6x + 12x^2 - 8x^3.$
107. $x^6 - 3x^4 + 3x^2 - 1.$

109. $m^3n^3 + 6m^2n^2a + 12mna^2 + 8a^3.$
110. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$

112. $4m^2 + 9n^2 + 1 - 12mn + 4m - 6n.$
113. $a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.$

Exercise 9

1. $a(x + y).$
2. $3(x + 3y).$
4. $5m(n - 2m).$
5. $8m(m - 2).$
7. $x(4x - y + 1).$
8. $3x(x - y + 2y^2).$
10. $(x + 4)(3 + a).$
11. $(m - n)(p + q).$
13. $(x^2 - x - 1)(a - b)$

14. $(x + 6y)(3a - 2b + 1).$
16. $4(2m + 3n)(m - n).$
17. $(4x - y)(2x - 2y + 3 + a).$
19. $(x + 4)(x - 4).$

20. $(1 + 5x)(1 - 5x).$
22. $8(a + 2b)(a - 2b).$

23. $2a(x + 3y)(x - 3y).$
25. $m(m + 4)(m - 4).$
26. $(2mn + 3c)(2mn - 3c).$
28. $(a - 1 + b)(a - 1 - b).$
29. $(a + b + 2)(a - b - 2).$
31. $(x - y + 3a)(x - y - 3a).$
32. $(2xy + z + 1)(2xy - z - 1).$
34. $(m + n + a - b)(m + n - a + b).$
35. $(2m + n - 2)(2m - n).$
37. $4(p + q)(x + 4)(x - 4).$
38. $(m + n)(m - n)(x + 7).$
40. 1820.
41. 400.
43. 1800.

Exercise 10

1. $(x+3)(x^2-3x+9)$. **2.** $(2+x)(4-2x+x^2)$. **4.** $(x-4)(x^2+4x+16)$.

5. $(1-2x)(1+2x+4x^2)$. **7.** $2(2a+b)(4a^2-2ab+b^2)$.
8. $(2a-3b)(4a^2+6ab+9b^2)$. **10.** $a(b+3a)(b^2-3ba+9a^2)$.

11. $3(ab-2)(a^2b^2+2ab+4)$.
13. $(x+y-z)(x^2+2xy+y^2+xz+yz+z^2)$.
14. $(x-2y+1)(x^2-4xy+4y^2-x+2y+1)$.
16. $(2x+2+y)(4x^2+8x+4-2xy-2y+y^2)$.
17. $(3-x+y)(9+3x-3y+x^2-2xy+y^2)$.
19. $(m-2n+3)(m^2+2mn-3m+4n^2-12n+9)$.
20. $(m+n-a+b)(m^2+2mn+n^2+am+an-bm-bn+a^2-2ab+b^2)$.

Exercise 11

1. $(x+2)^2$. **2.** $(x-3)^2$. **4.** $(4x-3)^2$.
5. $(1-4xy)^2$. **7.** $(3a+2b)^2$. **8.** $(4a+b)^2$.
10. $(a+2)(a+4)$. **11.** $(a+1)(a+6)$. **13.** $(m-1)(m-6)$.
14. $(m-3)(m-6)$. **16.** $(m+3)(m-4)$. **17.** $(m+5)(m-3)$.
19. $(a+3b)(a-7b)$. **20.** $(a+4b)(a-9b)$. **22.** $(2x+3)(x-1)$.
23. $(3x-2)(x+2)$. **25.** $(5x+1)(2x-3)$. **26.** $(2x+3)(2x-5)$.
28. $(3-4x)(7+x)$. **29.** $(3xy-8)(xy-1)$. **31.** $x(3x-2)(4x-1)$.
32. $x^2(x+6)(x+7)$. **34.** $(a^2+3)^2$. **35.** $(3-a^2)^2$.

37. $(a^2+1)(a+2)(a-2)$. **38.** $(3a^2+2)(a+2)(a-2)$.

40. $(a+2)^2(a-2)^2$. **41.** $(9+a^2)^2$. **43.** $(3x^2+2)(2x^2-3)$.

44. $(x+2)(x-2)(2x+1)(2x-1)$. **46.** $(m-n+1)^2$.
47. $(2m-n-2)^2$. **49.** $(m+n-2)(m+n+1)$.
50. $(3x-3y-1)(x-y-z)$. **52.** $(x+y+1)^2$.

53. $(x-y+2)^2$. **55.** $(x-y-3z)^2$. **56.** $(x+2)^3$.
58. $(a-6)^3$. **59.** $(5a+2)^3$.

Exercise 12

1. $(x+y)(a+4)$. **2.** $(x+3)(4+y)$. **4.** $(4x+3c)(x+3y)$.
5. $(x-y)(a+2c)$. **7.** $(3y-a)(3x+1)$. **8.** $(3x-a)(y+3a)$.

10. $2(1+m)(1-m)(2-m)$. **11.** $(m+1)(m-1)(am-3)$.
13. $(x-y-z)(a+b)$. **14.** $(2x+3y-z)(a+2)$.
16. $(m-2n)(m-2n-3)$. **17.** $(3m-1)(3m-1-a)$.
19. $(a+b)(a-b+5)$. **20.** $(a-2b)(a+2b-3)$.
22. $(2a+1)(a+2b-2)$. **23.** $(x+y)(x^2-xy+y^2+1)$.
25. $(a-b)(a^2+ab+b^2-a-b)$. **26.** $(a^2-ab+b^2)(a+b-1)$.
28. $(a-b+1)(a-b-1)$. **29.** $(m+3n+3)(m+3n-3)$.
31. $(m+3-2n)(m-3+2n)$. **32.** $(m-2n+4c)(m-2n-4c)$.
34. $(x-1+y+z)(x-1-y-z)$. **35.** $(3y+2x+3)(3y-2x-3)$.

37. $(3x-1+y+2z)(3x-1-y-2z)$.

38. $(x - y + a - b)(x - y - a + b)$.
40. $(8a^2 + 4a + 1)(8a^2 - 4a + 1)$.
41. $(x^2 + x - 1)(x^2 - x - 1)$.
43. $(2x^2 + 3b^2 + 2ab)(2a^2 + 3b^2 - 2ab)$.
44. $(x^2y^2 + xy + 1)(x^2y^2 - xy + 1)$.
46. $(5y^2 + y - 2)(5y^2 - y - 2)$.
47. $(7y^2 + y - 1)(7y^2 - y - 1)$.

Exercise 13

1. $(x^2 + 1)(x + 1)(x - 1)$.
2. $(4 + x^2)(2 + x)(2 - x)$.
4. $(25x^2 + 1)(5x + 1)(5x - 1)$.
5. $(9x^2 + 4y^2)(3x + 2y)(3x - 2y)$.

7. $(m + 1)(m^2 - m + 1)(m - 1)(m^2 + m + 1)$.
8. $(2 + m)(4 - 2m + m^2)(2 - m)(4 + 2m + m^2)$.

10. $(4m^2 + 1)(16m^4 - 4m^2 + 1)$.
11. $(m^2 + 4n^2)(m^4 - 4m^2n^2 + 16n^4)$.

13. $(x^2 + 2x + 2)(x^2 - 2x + 2)(x^2 + 2)(x^2 - 2)$.

14. $(2x^2 + 6x + 9)(2x^2 - 6x + 9)(2x^2 + 9)(2x^2 - 9)$.

16. $(x^2y^3 + 4)(x^2y^3 - 4)$.

17. $(x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$.

19. $(x^3 + y^2)(x^3 - x^3y^2 + y^4)$.
20. $(x^2 + y^4)(x^4 - x^2y^4 + y^3)$.

Exercise 14

1. $36x^3y^2$.
2. $180x^3y^2$.
4. $1260x^3y^2z^4$.

5. $4(x + 1)(x - 1)$.
7. $2(x + 2)(x - 1)(3x + 1)$.

8. $6(x + 2)(4x^2 - 1)$.
10. $(m + 1)(2m - 3)^2$.
11. $(m + n)(m - n)^2$.

13. $m(m + 1)(m - 1)$.
14. $6(mn + 2)(mn - 2)(2mn + 1)$.

16. $(x - 4)(4x + 1)(2x - 1)$.
17. $(x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$.

19. $4x(x + 4)(x - 4)(x^2 + 4x + 16)$.
20. $4x(x + 3)(x - 3)(x^2 + 1)$.
22. $4m(2m - 3n)^2(4m^2 + 6mn + 9n^2)$.
23. $(m^2 + n^2)(m + n)^2(m - n)$.

25. $(x + 2y - z)(x - 2y - z)(x + 2y + z)$.
26. $(4x - 3y)(3x - 4y)^2(9x^2 + 12xy + 16y^2)$.
28. $9(x + 4y)(x - 3y)(2x - y)(4x + y)(x - 2y)$.

Supplementary Problems

1. $x(x - 3y + 1)$.
2. $2ab(b - 2 + a)$.
4. $(3a + 1)(3a - 1)$.
5. $(2x + 3y)(2x - 3y)$.
7. $2(2m + n)(2m - n)$.
8. $3x(y + 2x)(y - 2x)$.

10. $(2 + a)(4 - 2a + a^2)$.
11. $(y - 4)(y^2 + 4y + 16)$.
13. $(2m + 3n)(4m^2 - 6mn + 9n^2)$.
14. $3a(p + 2q)(p^2 - 2pq + 4q^2)$.

16. $(1 - x - y)(1 + x + y + x^2 + 2xy + y^2)$.
17. $(x + y + 1)(x^2 + 2xy + y^2 - x - y + 1)$.

19. $(x-4)(x-5)$.
20. $(x+4)(x-6)$.
22. $(2-a)^2$.
23. $(mn+6)(mn+7)$.
25. $(2x+3)(x+2)$.
26. $(3a-4)(2a+3)$.

28. $(x^2+1)(x+2)(x-2)$.
29. $(x^3+2)(x-1)(x^2+x+1)$.
31. $(3x+y)^2$.
32. $(2a-5b)^2$.
34. $(a-b)(4-a)$.
35. $(y+2)(x-y)$.
37. $(x-y)(a-2b)(a^2+2ab+4b^2)$.
38. $a(4+b)(2-y)$.

40. $(x-y)(x+y+3)$.
41. $(x+2y)(x+2y+1)$.
43. $(x-2)(x+y+1)$.

44. $(m-n)(m^2+mn+n^2-m+n)$.
46. $(5+a-3)(5-a+3)$.
47. $(3x-y+4a)(3x-y-4a)$.
49. $(2x-y+z)(x-1)$.

50. $(x-1)^3$.
52. $(2a+3)^3$.
53. $(1-4x)^3$.

55. $(m+n-3)^2$.
56. $(x-2y+a+b)(x-2y-a-b)$.
58. $(4x^2+y^2)(2x+y)(2x-y)$.
59. $(5a^2+4)(5a^2-4)$.
61. $(2m^2+n^2)(4m^4-2m^2n^2+n^4)$.
62. $(m^2-y)(m^4+m^2y+y^2)$.
64. $(x^2+x+1)(x^2-x+1)$.
65. $(m^2+m+2)(m^2-m+2)$.

67. $(2x^2+3xy-3y^2)(2x^2-3xy-3y^2)$.

68. $(3x^2+5xy-y^2)(3x^2-5xy-y^2)$.
70. $(x+y+4)(x+y-1)$.
71. $(2x+y-3)(2x+y+2)$.
73. $(3m-3n-2)(3m-3n-1)$.

Exercise 15

1. $x=4$.
2. $x=1$, $x=-2$.
4. $x=\frac{1}{3}$.

5. $x=-\frac{4}{5}$.
16. $\dfrac{y}{3x}$.
17. $\dfrac{4b}{3ac}$.
19. $\dfrac{2}{x+2}$.
20. $\dfrac{x}{a}$.
22. $\dfrac{x-2}{x+2}$.
23. $\dfrac{a}{b}$.
25. 2.
26. $\dfrac{b+a}{b-a}$.
28. $\dfrac{2}{a}$.
29. $\dfrac{5x}{y}$.
31. $\dfrac{x-3}{x+1}$.
32. $-\dfrac{x+2}{4x+1}$.
34. $\dfrac{3-x}{3-2x}$.
35. $\dfrac{1}{a+b}$.
37. $\dfrac{x-y}{x}$.
38. $\dfrac{x-y+1}{x-y}$.
40. $\dfrac{y-1}{y+1}$.
41. $\dfrac{a-2}{a+2}$.
43. $\dfrac{(2c+1)^2}{4c^2-2c+1}$.
44. $\dfrac{9c^2+3c+1}{(3c-1)^2}$.
46. $\dfrac{2x-3y}{3x+2y}$.
47. $-\dfrac{x^2+1}{x^2+4y^2}$.
49. $\dfrac{2a-3b-4c}{ac}$.
50. $\dfrac{3x-y+2z}{xyz}$.

Exercise 16

1. $\frac{1}{2}$.
2. $\frac{10}{13}$.
4. $\frac{4}{3}$.

5. $6b$.
7. $\dfrac{x}{12}$.
8. 4.

10. $\dfrac{bx}{6}$.
11. $\dfrac{5a}{2bx}$.
13. $\dfrac{a^3}{18b^2}$.

14. $\frac{4y^2}{xz}$.

16. $\frac{x}{y-1}$.

17. $\frac{1}{2}$.

19. $\frac{a^2-ab+b^2}{x^2-xy+y^2}$.

20. $-\frac{x+1}{(x^2+x+1)(a+1)}$.

22. $\frac{x+3}{x+6}$.

23. $\frac{x+y}{7}$.

25. $\frac{1}{a^2-2a+4}$.

26. $\frac{(x+2)(x+7)}{(x-2)(x-3)}$.

28. $\frac{2x+3}{5x-1}$.

29. $x-2$.

31. $\frac{y(x^2+y^2-xy)}{2x+y}$.

32. $\frac{c-2d}{c+2d}$.

34. 1.

35. $\frac{(a-2)(3a+5)}{(a+2)(a+5)}$.

37. 1.

38. $\frac{n(2m-3)}{m+n+1}$.

40. $\frac{x^2+xy+y^2+2}{x+y}$.

Exercise 17

1. $\frac{5x}{20x^2}, \frac{12}{20x^2}$.

2. $\frac{10}{6xy}, \frac{9}{6xy}$.

4. $\frac{1}{x-1}, \frac{-2}{x-1}$.

5. $\frac{4(x+2)}{(x+2)(x-2)^2}, \frac{-5(x-2)}{(x+2)(x-2)^2}$.

7. $\frac{2(x+1)}{x(x^2-1)}, \frac{x^2}{x(x^2-1)}, \frac{3(x-1)}{x(x^2-1)}$.

8. $\frac{x(3x+1)}{9x^2-1}, -\frac{5(3x+1)}{9x^2-1}, \frac{2x}{9x^2-1}$.

10. $\frac{4(x^2-1)}{(x+1)(x-1)^3}, -\frac{x+1}{(x+1)(x-1)^3}, \frac{2(x-1)^2}{(x+1)(x-1)^3}$.

11. $\frac{x^2+2x-3}{3}$.

13. $\frac{1}{x-3}$.

14. $\frac{10+13x+6x^2}{(x-6)(x+5)}$.

16. x.

17. $\frac{137}{24}$.

19. $-\frac{59}{24}$.

20. $\frac{19a}{50}$.

22. $\frac{25a-6}{24}$.

23. $\frac{a+39}{(a-9)(a+3)}$.

25. $\frac{a^2-3a+15}{a(a-5)}$.

26. $\frac{2y+3}{y^2-9}$.

28. $\frac{4-y^2}{1+y}$.

29. $\frac{19y-6y^2-12}{3y-5}$.

31. $\frac{2a^3-4a^2+3a-1}{a^2+1}$.

32. $\frac{m^2+5m+30}{(m-3)(m+6)}$.

34. $\frac{3m^2-4m+2}{m(m-1)}$.

35. $\frac{m^2-16m-8}{(m+2)(m-2)^2}$.

37. $\frac{3-3m-4m^2}{m(m^2-1)}$.

38. $\frac{6x^2+23x+12}{(x+2)(2x-3)(2x+3)}$.

40. $\frac{x^3+8x^2-12x-7}{(x+1)(x-1)(x-2)}$.

41. $\frac{20-10x^2-x^3}{(x+2)(x-1)(x-2)}$.

43. $\frac{x}{2x-3}$.

44. $\frac{-2}{x+y}$.

46. $\frac{3x^2-6x-18}{(2x-1)(x+1)(2x+3)(x-2)}$.

47. $\frac{6y+11}{(y-1)(y+3)}$.

49. $\frac{-m^2+m-9}{(m-1)(m-2)(2m+7)}$.

50. $\frac{5x^2-27}{(x^2-1)(x^2-3)(x+3)}$.

52. $\frac{a^3}{(a^3+b^3)(a^2+ab+b^2)}$.

Exercise 18

1. $\frac{1}{21}$.
2. $\frac{20}{57}$.
4. $\frac{2x + 3}{12x^2}$.
5. $\frac{20 - x}{25}$.
7. $\frac{1 - x}{1 + x}$.
8. $\frac{2(x + 1)}{x + 2}$.
10. $\frac{x(2x^2 + 1)}{6(2x - 1)}$.
11. $\frac{x + 2}{x + 4}$.
13. $\frac{2(5x - 4)(2x + 1)}{x - 2}$.
14. $\frac{x(x - 2)}{(x - 1)(x + 2)}$.
16. $\frac{a + 5x}{5a - x}$.
17. $\frac{27m^2 - 17}{(3m - 2)(15m + 7)}$.
19. $\frac{x^2 - 2xy + y^2 + 1}{4xy}$.
20. $\frac{(3x - 10)(x + 2)}{(2x + 3)(x - 2)}$.
22. $\frac{x - 3}{x + 1}$.
23. $\frac{3x^2 + 1}{3x}$.
25. $2(a^2 - 2a + 4)$.
26. $\frac{3x - 1}{5x + 2}$.
28. $\frac{-1}{x + y}$.
29. $\frac{(m + 1)(m^2 + 1)}{(m - 1)(m^2 - 3)}$.
31. $\frac{2(x - 1)}{3(x + 1)}$.
32. $\frac{1 + x}{2(2 - x)}$.
34. $\frac{3x - 1}{2x - 1}$.
35. $\frac{x^2 + x + 1}{x}$.

Exercise 19

7. 3.
8. -1.
10. -1.
11. 5.
13. 1.
14. 2.
16. -2.
17. 5.
19. 6.
20. -1.
22. 0.
23. 1.
25. 0.
26. $\frac{8}{11}$.
28. 4.
29. -2.
31. $\frac{59}{67}$.
32. 4.
34. 3.
35. $-\frac{1}{2}$.
37. -12.
38. $-\frac{17}{21}$.
40. $\frac{17}{9}$.
41. $-\frac{3}{5}$.
43. $-\frac{9}{4}$.
44. $-\frac{1}{6}$.
46. $\frac{1}{8}$.
47. $\frac{1}{8}$.
49. $\frac{a - 2y + 4b}{2}$.
50. $\frac{5a + 7}{6}$.
52. $5a - 1$.
53. $\frac{10 - 4a - b}{4}$.
55. $\frac{2b - 11}{a - 3}$.
56. $\frac{4 + 2b}{10 - 3a}$.
58. $\frac{ay - 2a + 4}{2}$.
59. $\frac{3b - 3}{4 - a}$.
61. $\frac{a + 10}{a - 1}$.
62. $\frac{3a - 7}{b - c + 3}$.
64. $\frac{y + 5}{9y - 13}$.
65. $\frac{2y - 7}{15(y + 1)}$.
67. $\frac{8}{7}$.
68. $-\frac{8}{7}$.
70. $\frac{17}{4m}$.
71. $\frac{-2}{7a}$.
73. $2 - y - \frac{4}{y}$.
74. $\frac{5 - 2y^2}{y}$.
76. $\frac{3y - 2}{4y + 3}$.

Exercise 20

1. 3.
2. $\frac{3}{5}$.
4. $\frac{2}{3}$.
5. $-\frac{1}{2}$.
7. No solution.
8. $\frac{1}{4}$.

10. $\frac{3}{4}$.
11. -4.
13. No solution.
14. $\frac{1}{5}$.
16. No solution.
17. No solution.
19. $\frac{6-a}{6+a}$.
20. $\frac{a}{9}$.
22. $\frac{2a-1}{a}$.
23. $\frac{1}{4}$.
25. -4.
26. -4.
28. $\frac{9}{16}$.
29. 3.
31. $-\frac{13}{21}$.

Exercise 21

1. 47, 94.
2. 76.
4. 34, 76.
5. 76, 38, 57.
7. 50°, 65°, 65°.
8. 30°, 60°, 90°.
10. 13 by 17.
11. 4, 10.
13. 39 dimes, 31 quarters.

14. 48 of the $20 prizes and 208 of the $5 prizes.

16. 120 lb.
17. 16 oz.
19. 10 mi.
20. 1120 mi.
22. 98.
23. 89.
25. 36, 82.
26. 15, 35.
28. 58, 14.
29. $\frac{6}{14}$.
31. 15 mph.
32. $1\frac{5}{7}$ mph.
34. $2\frac{8}{11}$ days.
35. $2\frac{74}{103}$ days.
37. $4\frac{10}{11}$ days.

38. $5\frac{101}{125}$ days.
40. 4, 6.
41. $9,000 at $3\frac{1}{2}$% and $11,000 at 4%.
43. $16\frac{4}{5}$ hr.

44. 72 hr.
46. $1\frac{4}{7}$ hr and $2\frac{1}{7}$ hr.
47. $1\frac{5}{7}$ qt.

Exercise 22

1. $-13, -3, 17, -\frac{1}{2}, -\frac{27}{4}$.
2. $-1, 1, -1, -\frac{1}{4}, -\frac{1}{4}$.
4. $0, 0, \frac{50}{9}, \frac{56}{9}, -14, -6$.
5. $0, 0, 0, -\frac{9}{8}, -\frac{20}{27}, 90$.

7. $4x^2 - 1, x^4 - 1, t^2 + 2t, \frac{1}{4}t^2 - 1, t^4 + 2t^2$.
8. $-1, -3, \frac{x^2+1}{x^2-1}, \frac{1+x}{1-x}, \frac{1+x^2}{1-x^2}$.

10. $9, -16, 5x + 1, 26, 35, 3$.
11. $x - 1, 4x^3 + 8x^2 - 3x - 9$.

13. $6 + h, 2x + h$.
14. $\frac{-1}{4(4+h)}, \frac{-1}{x(x+h)}$.

16. $y = \frac{x-5}{2}; x = 2y + 5$.
17. $y = \frac{x-1}{x+1}; x = \frac{1+y}{1-y}$.

19. $A = \frac{1}{2}bh$.
20. $v = x^3$.

Exercise 24

10. $y = 2x - 5$.
11. $y = \frac{x-4}{2}$.
13. $y = -x$.
14. $y = \frac{4x-1}{2}$.
16. $y = 2x$.
17. $y = -\frac{1}{3}x$.
19. 0.
20. 0.
22. $-1.6, 1.6$.
23. $-1, 4$.
25. $0.6, -3.1$.
26. 0.

Exercise 26

1. $x = 4, y = 2.$
2. $x = 3, y = -1.$
4. $x = -2, y = 3.$
5. $x = -1, y = -1.$
7. $x = 5, y = -1.$
8. $x = 1, y = 1.$
10. Inconsistent.
11. Dependent.
13. $x = -\frac{3}{5}, y = -\frac{11}{5}.$
14. $x = -\frac{1}{2}, y = \frac{5}{2}.$
16. $x = \frac{5}{3}, y = 0.$
17. $x = \frac{3}{2}, y = \frac{3}{2}.$
19. Inconsistent.
20. $x = \frac{9}{2}, y = -\frac{7}{2}.$

Exercise 27

1. $x = 1, y = -3.$
2. $x = -\frac{4}{3}, y = \frac{2}{9}.$
4. $x = -4, y = -3.$
5. $x = 3, y = -2.$
7. $x = \frac{5}{4}, y = 2.$
8. $x = -1, y = 2.$
10. $x = 4, y = -6.$
11. $x = -2, y = 0.$
13. $x = \frac{5}{13}, y = -5.$
14. $x = \frac{2}{3}, y = -\frac{3}{2}.$
16. $x = \frac{3}{2}, y = -3.$
17. $x = -1, y = -\frac{1}{2}.$
19. $x = 1, y = -1.$
20. $x = \frac{15}{14}, y = -\frac{10}{3}.$
22. $x = 2, y = -\dfrac{2}{a}.$
23. $x = b, y = -a.$
25. $x = \dfrac{3}{a+b}, y = \dfrac{b-2a}{a+b}.$
26. $x = \dfrac{a+b}{a}, y = \dfrac{a-b}{b}.$
28. $x = \dfrac{b+d}{ad+bc}, y = \dfrac{a-c}{ad+bc}.$
29. $x = \dfrac{a+b}{b}, y = \dfrac{a-b}{a}.$
31. $x = \dfrac{ce-bf}{ae-bd}, y = \dfrac{af-cd}{ae-bd}.$
32. $x = a - b, y = b - a.$

Exercise 28

1. $x = 1, y = 2, z = 3.$
2. $x = -1, y = 2, z = 1.$
4. $x = 3, y = -1, z = 2.$
5. $x = 2, y = 0, z = -3.$
7. $u = 3, v = 0, w = 0.$
8. $u = 0, v = -1, w = -1.$
10. $u = 3, v = 2, w = -4.$
11. $u = \frac{1}{2}, v = \frac{1}{3}, w = \frac{1}{4}.$
13. $x = a, y = a, z = -a.$
14. $x = \dfrac{1}{a}, y = \dfrac{1}{b}, z = \dfrac{1}{c}.$
16. $u = 2, v = 3, w = -1.$
17. $u = -\frac{1}{3}, v = \frac{1}{2}, w = -\frac{1}{4}.$
19. $x = -\frac{1}{4}, y = \frac{1}{3}, z = \frac{1}{2}.$
20. $x = \frac{3}{2}, y = \frac{4}{3}, z = \frac{2}{3}.$
22. $x = \dfrac{3}{a}, y = -\dfrac{1}{b}, z = \dfrac{4}{c}.$
23. $x = \frac{11}{3}, y = -1, z = -\frac{1}{3}.$
25. $u = \frac{5}{8}, v = \frac{3}{4}, w = -\frac{1}{8}.$
26. $u = \frac{5}{13}, v = \frac{8}{13}, w = -\frac{7}{13}.$
28. $x = \frac{2}{9}, y = \frac{1}{18}, z = \frac{2}{3}.$
29. $x = \frac{3}{20}, y = \frac{3}{10}, z = \frac{7}{20}.$

Exercise 29

1. 34 and 64.
2. 14 and 27.
4. 4 by 13 and 6 by 16.
5. $\frac{5}{8}$.
7. $8\frac{1}{2}$ and $2\frac{1}{2}$ mph.
8. 30 and 210 mph.
10. A, $13\frac{1}{11}$ days; B, $28\frac{4}{5}$ days; C, $20\frac{4}{7}$ days.
11. A, 15 hr; B, 60 hr; C, 30 hr.
13. $28\frac{4}{7}$ lb of 85¢ coffee and $71\frac{3}{7}$ lb of 92¢ coffee.
14. $18\frac{14}{27}$ gal of cream and $481\frac{13}{27}$ gal of milk.
16. 14 and 42.
17. 20 and 40.
19. \$2000 at $2\frac{1}{2}\%$, \$6000 at 3%, \$4000 at $3\frac{1}{2}\%$.
20. 80 lb and 100 lb.
22. 16 ft.
23. 100 games played, 45 won.
25. 5, 11, and 17.

Exercise 30

1. 15.
2. 43.
4. 2.
5. $-18a$.
7. 24.
8. 4.
10. 47.
11. -50.
13. 0.
14. 7.

Exercise 32

1. 6.
2. $\frac{1}{7}$.
4. $\frac{2}{3}$.
5. 8.
7. $\frac{5}{36}$.
8. $\frac{4}{27}$.
10. 6.
11. $\frac{9}{5}$.
13. 42.
14. 30.
16. $-22\frac{1}{2}$.
17. 22.
19. 25.
20. $\frac{25}{6}$.
22. $\frac{81}{7}$.
23. 0.9.
25. 45.
26. 3, $-$ 3.
28. 22.
29. $-\frac{2}{3}$.
31. 27 and 45.
32. \$25.05 and \$58.45.
34. 120 mi.
35. 3.2 mi.
37. $13\frac{1}{3}$ ft.
38. $4\frac{2}{3}$ in., 6 in., $8\frac{2}{3}$ in., 9 to 4.
40. 4 to 25.
41. Areas 121 to 9, volumes 1331 to 27.

Exercise 33

1. $S = kr^2$.
2. $A = kbh$.
4. $W = \dfrac{kx^3}{y^2}$.
5. $A = \dfrac{k}{d^2}$.
7. $V = \dfrac{kx}{y^2z}$.
8. $Z = \dfrac{xy}{5}$.
10. $V = \pi r^2h$.
11. $W = \dfrac{60}{x}$.
13. 29.4.
14. 3.
16. $533\frac{1}{3}$.
17. 160.
19. 144 ft, 256 ft, 1600 ft.
20. 14,175 lb.
22. 64 lb, 25 lb.
23. Reduced 50%.
25. 14 days.
26. 1.75 sec, 3.51 sec, 0.70 sec.
28. 200 ohms.
29. Ratio 10 to 1.
31. 1 to 4.

Exercise 34

1. 3^5.
2. 4^6.
4. 3^6.
5. 4^{2n}.
7. $6a^5$.
8. $12x^5y^2$.
10. $18a^4b^3$.
11. $-27a^5$.
13. $8x^6y^9$.
14. $9a^2b^6$.
16. $36c^{10}$.
17. x^9y^9.
19. 2^4a^4.
20. $\dfrac{8}{a^3b^3}$.
22. $\dfrac{2x^4}{3y}$.
23. $\dfrac{a^3}{2b}$.
25. $\dfrac{x^2z}{y}$.
26. $\dfrac{x^5y^5}{2z^5}$.
28. $\dfrac{27x^9y^6}{8}$.
29. $\dfrac{a^4b^4}{16c^8}$.
31. $\dfrac{p^{10}r^5}{q^5}$.
32. $\dfrac{p^8}{16q^4r^4}$.
34. $\dfrac{3x^7}{4y^3}$.
35. $\dfrac{1}{40x^7y^{12}}$.

37. $\frac{a^4}{c^2}$. **38.** $\frac{x^4yz}{4}$. **40.** x^n.
41. 2^{n+1}. **43.** 5^nx^2. **44.** $\frac{a}{2^{2n}}$.
46. $\frac{y}{24x^{11}}$. **47.** $\frac{x^5}{a^2y^{10}b}$. **49.** $\frac{y^{18}z^6}{x^4}$.
50. $\frac{2x^{10}}{3y^5z^{14}}$.

Exercise 35

1. $\frac{1}{16}$. **2.** 1. **4.** $\frac{1}{2}$.
5. 1. **7.** 5. **8.** 6.
10. $\frac{1}{4}$. **11.** $\frac{1}{16}$. **13.** $\frac{27}{32}$.
14. $\frac{1}{9}$. **16.** -8. **17.** $\frac{1}{8}$.
19. $\frac{9}{2}$. **20.** 25. **22.** $\frac{1}{8}$.
23. 25. **25.** $\frac{25}{64}$. **26.** $\frac{9}{16}$.
28. $\frac{1}{y}$. **29.** $\frac{1}{y^3}$. **31.** $\frac{a}{bc^2}$.
32. $\frac{a^2}{c}$. **34.** x^3y. **35.** $\frac{2x^2y}{3}$.
37. $18pq$. **38.** $\frac{2p^2q^5}{5}$. **40.** $\frac{b^2}{3a}$.
41. $\frac{9c^6}{4a^4b^2}$. **43.** $\frac{p^4q^6}{r^2}$. **44.** $\frac{2pq^3}{3r^4}$.
46. $\frac{b^8}{a^4}$. **47.** $\frac{b^9}{a^3y^{15}}$. **49.** $\frac{17}{24}$.
50. $\frac{5}{8}$. **52.** $y^2 - x^2$. **53.** $\frac{1}{x^3 + y^3}$.

Exercise 36

1. 5. **2.** 4. **4.** 4.
5. 81. **7.** -32. **8.** -5.
10. $\frac{1}{16}$. **11.** $\frac{1}{8}$. **13.** $\frac{32}{243}$.
14. $\frac{3}{2}$. **16.** 5. **17.** 0.01.
19. $\frac{9}{10}$. **20.** $\frac{16}{81}$. **22.** x^2.
23. $a^{5/6}$. **25.** $\frac{1}{x^{7/6}}$. **26.** $\frac{1}{a^{1/3}}$.
28. $\frac{1}{x^{3/4}}$. **29.** $a^{17/6}$. **31.** $\frac{125}{a^2}$.
32. $\frac{c^{3/2}}{8}$. **34.** $2^{1/2}a$. **35.** $\frac{1}{a^{3/2}b}$.
37. $\frac{x^2}{125}$. **38.** $\frac{y^{7/2}}{16}$. **40.** $12x^{2/3}y^{1/3}$.
41. $\frac{10b^{1/2}}{a^{3/4}}$. **43.** $\frac{12}{a^{5/6}b^{1/6}}$. **44.** $8xy^{1/3}$.
46. $x^{5/6}y^{1/2}$. **47.** $\frac{27b^{11/6}}{8a^{5/6}}$. **49.** $\frac{x^{5/2}}{y}$.

50. $x^{10}y^{7/2}$.
52. $\dfrac{3ab^{11/12}}{4}$.
53. $\dfrac{4b^{5/6}}{9a^{5/3}}$.
55. $x - 1$.
56. $x - 2x^{1/2}y^{1/2} + y$.
58. $2x + 5x^{1/2}y^{1/2} - 12y$.
59. $x - y$.
61. $x + y$.

Exercise 37

1. $\sqrt[3]{a^2}$.
2. $\sqrt{5^3}$.
4. $3\sqrt{a}$.
5. $\sqrt{2x^5}$.
7. $-\sqrt[3]{x^2}$.
8. $\sqrt[3]{x^2}$.
10. $a\sqrt[5]{b^3}$.
11. $\sqrt[3]{-2a^2}$.
13. $\sqrt{5^3(x + y)}$.
14. $a\sqrt[3]{(x - y)^2}$.
16. $a^{3/2}$.
17. $2^{1/3}a^{1/3}$.
19. $2^{1/3}a^{2/3}$.
20. $6^{1/4}a^{3/4}b^{1/4}$.
22. $2^{1/2}(x + y)^{5/2}$.
23. $(x - y)^{1/3}$.
25. $-a^{3/5}b^{4/5}$.
26. $7^{1/2}x^{1/2}y^{1/2}$.
28. $5^{1/6}$.
29. $3^{1/2}a^{1/4}$.
31. $\sqrt{18}$.
32. $\sqrt[3]{24}$.
34. $\sqrt{12a^2b}$.
35. $\sqrt[n]{2^n \cdot 3}$.
37. $\sqrt{2x^3}$.
38. $\sqrt{2ab}$.
40. $\sqrt{4 - x}$.
41. $\sqrt[3]{8x^3 - 8}$.
43. $\sqrt{3z(x + y)}$.
44. $\sqrt{\dfrac{a - b}{a + b}}$.
46. $2\sqrt{2}$.
47. $2\sqrt{5}$.
49. $2ab^2\sqrt{b}$.
50. $3ab\sqrt{3a}$.
52. $-2\sqrt[3]{2b}$.
53. $3xy^2\sqrt[3]{3x^2}$.
55. $2bc\sqrt[4]{2b}$.
56. $3y\sqrt[4]{x^3y}$.
58. $\frac{1}{3}\sqrt{3}$.
59. $\frac{1}{2}\sqrt{6}$.
61. $\frac{1}{3}\sqrt[3]{9}$.
62. $-\frac{1}{2}\sqrt[3]{6}$.
64. $\dfrac{1}{5a}\sqrt{15}$.
65. $\dfrac{1}{3x}\sqrt{6}$.
67. $\dfrac{a^2b}{2c}\sqrt{14bc}$.
68. $\dfrac{1}{5z}\sqrt[3]{90x^2y^2z^2}$.
70. $\dfrac{5x^2}{3yz}\sqrt{2xy}$.
71. $\dfrac{x^2}{6y^2z}\sqrt{30yz}$.
73. $\sqrt{5}$.
74. $\sqrt{3a}$.
76. $ab^2\sqrt{2a}$.
77. $xy\sqrt{2y}$.
79. $z\sqrt[3]{xy^2}$.
80. $yz\sqrt[6]{x^4z^3}$.
82. $\frac{1}{3}\sqrt{6a}$.
83. $\dfrac{2a}{5}\sqrt{10a}$.
85. $\sqrt{a}$.
86. $\sqrt[3]{x}$.
88. $b\sqrt[10]{81a^5}$.
89. $\sqrt[5]{3x^2y^2}$.

Exercise 38

1. $-2\sqrt{3}$.
2. $6\sqrt{2}$.
4. $-2\sqrt{6}$.
5. $\frac{17}{6}\sqrt{6}$.
7. $\frac{22}{15}\sqrt{15}$.
8. $-\frac{1}{2}\sqrt{10}$.
10. $(2b - 4a + 1)\sqrt{ab}$.
11. 0.
13. $-9\sqrt[3]{5}$.
14. $(2a + 2)\sqrt[3]{3a}$.
16. $(a + b + 1)\sqrt{a + b}$.
17. $(x - y - 1)\sqrt{x - y}$.
19. $\frac{4}{9}\sqrt{3} + \frac{1}{3}\sqrt[3]{9}$.
20. $\dfrac{x^2 + y^2}{x^2 - y^2}\sqrt{x^2 - y^2}$.
22. $\dfrac{x - y + 2}{2(x - y)}\sqrt{2(x - y)}$.
23. $(3y + 1)\sqrt{6xy} - 2x\sqrt[3]{6xy}$.
25. $(6a^2 - 3)\sqrt{2a} - 2a\sqrt[3]{2a}$.
26. $(2y + 1)\sqrt{2xy}$.
28. $(2x + y + 1)\sqrt[3]{2xy}$.

Exercise 39

1. $\sqrt{6}$.
2. $5\sqrt{2}$.
4. $6x\sqrt{3}$.
5. $2y\sqrt{x}$.
7. $2\sqrt{3}$.
8. $\frac{55}{3}$.
10. 20.
11. $50a$.
13. -24.
14. 16.
16. $3\sqrt[3]{a^2}$.
17. $4a$.
19. $-2a\sqrt[3]{2}$.
20. $24\sqrt[3]{4}$.
22. $4xy$.
23. $2xy$.
25. $\sqrt[6]{32}$.
26. $\sqrt[4]{27}$.
28. $\frac{1}{3}\sqrt[6]{243}$.
29. $\frac{1}{3}\sqrt[6]{1215}$.
31. $\sqrt[12]{3^7}$.
32. $\sqrt[4]{18}$.
34. $5 + 2\sqrt{6}$.
35. $7 - 4\sqrt{3}$.
37. -1.
38. 1.
40. $70 - 12\sqrt{5}$.
41. $44 + 5\sqrt{6}$.
43. $30 - 14\sqrt{3}$.
44. $-4 - 4\sqrt{5}$.
46. $14 - 2\sqrt{10} + 2\sqrt{14} - 2\sqrt{35}$.

Exercise 40

1. $\sqrt{3}$.
2. $\frac{1}{3}\sqrt{6}$.
4. $\sqrt{13}$.
5. $\frac{1}{3}\sqrt{15}$.
7. $\frac{1}{2x}\sqrt[3]{14x^2y}$.
8. $\frac{3}{5}\sqrt{5}$.
10. $\frac{4}{15}\sqrt[3]{15}$.
11. $\frac{1}{x}\sqrt[4]{2x^3}$.
13. $\frac{5}{8}\sqrt{3}$.
14. $\frac{3}{4}\sqrt[4]{24}$.
16. $\sqrt[6]{3}$.
17. $\frac{1}{2}\sqrt[6]{112}$.
19. 2.
20. $\sqrt[12]{162}$.
22. $\dfrac{3\sqrt{2} + \sqrt{3}}{3}$.
23. $2\sqrt{3} - 3\sqrt{7}$.
25. $\sqrt{3} + \sqrt{2}$.
26. $\dfrac{27\sqrt{2} + 8\sqrt{3}}{6}$.
28. $-\dfrac{1 + \sqrt{3}}{2}$.
29. $\dfrac{2 - \sqrt{2}}{2}$.
31. $2(\sqrt{5} + \sqrt{3})$.
32. $\dfrac{3\sqrt{2} + 2}{7}$.
34. $\dfrac{2\sqrt{6} - 2}{5}$.
35. $6 - 2\sqrt{6}$.
37. $\dfrac{12 + 7\sqrt{2}}{2} = 10.949$.
38. $\dfrac{4(\sqrt{3} + \sqrt{2}) - \sqrt{6} - 16}{14} = -0.419$.
40. $-\dfrac{1 + 4\sqrt{6}}{10} = -1.080$.
41. $\dfrac{-1 + \sqrt{35}}{2} = 2.458$.
43. $\dfrac{2 + x + 2\sqrt{2x}}{2 - x}$.
44. $\dfrac{x + 2\sqrt{xy} + y}{x - y}$.
46. $\dfrac{\sqrt{6} - 3}{3}$.
47. $\dfrac{5 + 3\sqrt{3}}{2}$.
49. $\dfrac{2\sqrt{3} + 3\sqrt{2} + \sqrt{30}}{6}$.
50. $\dfrac{\sqrt{15} - \sqrt{6}}{3}$.

Exercise 41

1. -5, $\sqrt{2}$, and $\sqrt[3]{-4}$ are real numbers; $2 - i$ and $2 + \sqrt{-5}$ are imaginary numbers; $3i$ and $\sqrt{-2}$ are pure imaginary numbers.

2. $3 + 3i$.
4. $5 + 5i$.
5. $10i$.
7. $90i$.
8. $3i\sqrt{2}$.
10. $-xy^2i$.
11. $2 - 3i\sqrt{5}$.
13. $3 + 12i\sqrt{3}$.
14. $9i$.
16. $-3i\sqrt{2}$.
17. $i\sqrt{3}$.
19. $2 - 3i$.
20. $17 - 10i$.
22. $2 + i$.
23. $17 - 51i$.
25. $13i$.
26. $-7 + i$.
28. $7 - 24i$.
29. $x^2 - y^2 - 2xyi$.
31. $2 + i$.
32. 1.
34. $\dfrac{3 - 6i}{5}$.
35. $1 - 2i$.
37. i.
38. $\dfrac{4 - 7i}{5}$.
40. $12 - i\sqrt{3}$.
41. $14 - i\sqrt{2}$.
43. $\dfrac{7 - 11i\sqrt{3}}{4}$.
44. $\dfrac{36 - i\sqrt{2}}{22}$.
46. -36.
47. $-2i\sqrt{15}$.
49. $10, -4i, 29$.
50. $2a, 2bi, a^2 + b^2$.

Supplementary Problems

1. $-a^{11}b^{13}$.
2. $12x^9$.
4. x^7.
5. a^{2m+1}.
7. 4^{3-n}.
8. 3^{2n+4}.
10. $\dfrac{1}{x^6} - \dfrac{1}{y^6}$.
11. $\frac{1}{16} - \frac{1}{81}$.
13. $3x^2y^3$.
14. $4a^2bc$.
16. $\dfrac{5x^2}{y}$.
17. $\dfrac{x^5y^6}{3}$.
19. $\dfrac{5y^8}{2xz^2}$.
20. $\dfrac{2x^2y^2}{3z^4}$.
22. $6r^5t^2$.
23. $\dfrac{rt}{72}$.
25. $\dfrac{8a^4}{9b^5}$.
26. $\dfrac{8p^5q^2}{9}$.
28. $\dfrac{4y^4}{9x^6}$.
29. $\dfrac{m^9n^6}{y^3}$.
31. $\dfrac{a^5}{b^9c^{13}}$.
32. $\dfrac{27a^5}{4b^9}$.
34. $\frac{9}{2}$.
35. 6.
37. $-\dfrac{(x - y)^2}{xy}$.
38. $\dfrac{b^3 + a^3}{ab(b^2 + a^2)}$.
40. $a^{1/6}$.
41. $a^{1/6}$.
43. $\dfrac{1}{x^2y^3}$.
44. $\dfrac{8}{x^4y^8}$.
46. $9x^2y$.
47. $\dfrac{4y}{x^2}$.
49. $\dfrac{b^9}{125a^4}$.
50. $\dfrac{a^3}{16b}$.
52. $4a$.
53. $\dfrac{1}{9b^{7/3}}$.
55. $\dfrac{81x^{1/2}}{16}$.
56. $\dfrac{x^{20}}{y^2}$.
58. $\dfrac{8x}{y^{3/2}}$.
59. $\dfrac{3^{1/2}x^{5/16}}{y}$.
61. $2 + \dfrac{x^{1/2}}{y^{1/2}} + \dfrac{y^{1/2}}{x^{1/2}}$.
62. $\dfrac{1}{x^{3/2}y^{3/2}} - x^{3/2}y^{3/2}$.

64. $\dfrac{9}{x^{1/3}} - 4x^{1/3}$.
65. $\dfrac{a}{4b}\sqrt{10a}$.
67. $-\dfrac{2x}{3y}\sqrt[3]{6xy^2}$.
68. $\dfrac{a}{2b^2}\sqrt[4]{200a^3}$.
70. $\dfrac{-2}{3xy}\sqrt[3]{x}$.
71. $\dfrac{-1}{ay^3}\sqrt[5]{2x^4a^4y^4}$.
73. $\dfrac{1}{a^2 - b^2}\sqrt{a^4 - b^4}$.
74. $x^2y\sqrt{2x}$.
76. xy^2.
77. $5\sqrt{2}$.
79. $\sqrt{6}$.
80. 0.
82. $3\sqrt[3]{4}$.
83. $-\frac{11}{18}\sqrt{3}$.
85. $6y\sqrt{2xy}$.
86 . $-5x^2\sqrt{3x}$.
88. $\frac{5}{2}\sqrt{2ab}$.
89. $\dfrac{xy^2 + y + 2x}{2y^3}\sqrt{2xy}$.

Exercise 42

1. $\log_2 16 = 4$.
2. $\log_5 25 = 2$.
4. $\log_2 \frac{1}{8} = -3$.
5. $\log_6 \frac{1}{6} = -1$.
7. $\log_{10} 10 = 1$.
8. $\log_3 3 = 1$.
10. $\log_{36} 6 = \frac{1}{2}$.
11. $\log_{64} 4 = \frac{1}{3}$.
13. $\log_9 \frac{1}{3} = -\frac{1}{2}$.
14. $\log_8 \frac{1}{4} = -\frac{2}{3}$.
16. $\log_{2/3} \frac{9}{4} = -2$.
17. $\log_{3/2} \frac{9}{4} = 2$.
19. $2^2 = 4$.
20. $6^2 = 36$.
22. $9^{1/2} = 3$.
23. $8^{5/3} = 32$.
25. $10^1 = 10$.
26. $4^0 = 1$.
28. $3^{-2} = \frac{1}{9}$.
29. $2^{-3} = \frac{1}{8}$.
31. $(\frac{1}{3})^{-3} = 27$.
32. $(\frac{1}{2})^{-5} = 32$.
34. 4.
35. 2.
37. $\frac{1}{2}$.
38. $\frac{1}{3}$.
40. 1.
41. 1.
43. 0.
44. −2.
46. 100.
47. 5.
49. 2.
50. 10.
52. $\frac{1}{16}$.
53. 9.
55. 1.
56. 64.
58. 5.
59. 4.
61. 81.
62. $\frac{1}{2}$.
64. $\frac{1}{27}$.
65. $\frac{1}{8}$.

Exercise 43

1. log 852 + log 561 + log 473.
2. 3 log 78 + log 7.12.
4. log 83 − log 51 − log 47.
5. log 14 + log 33 − log 28 − log 61.
7. $\frac{1}{2}$ log 57 + $\frac{1}{3}$ log 91 − 4 log 18.4.
8. $\frac{1}{2}$(log 28 + log 44 + log 21 − log 978).
10. log 85 + $\frac{1}{3}$ log 67 − 2 log 43 − $\frac{1}{2}$ log 65.
11. 0.9030.
13. 0.7781.
14. 1.6811.
16. 0.3597.
17. 0.7269.
19. 0.4203.
20. 1.5050.
22. 2.1582.

Exercise 44

1. 1.
2. 1.
4. 3.
5. 2.
7. −1.
8. −2.
10. −4.
11. −5.
13. 3.
14. −2.
16. 863.4.
17. 8.634.
19. 0.08634.
20. 0.8634.
22. 86340.
23. 0.08634.
25. 0.0008634.
26. 0.00008634.

Exercise 45

1. 0.4969.
2. 1.7574.
4. 4.7604.
5. 0.9586.
7. 7.9031 − 10.
8. 8.7634 − 10.
10. 2.8808.
11. 2.9542.
13. 9.5911 − 10.
14. 5.7604.
16. 15.3
17. 309.
19. 9.35.
20. 54000.
22. 0.475.
23. 0.0448.
25. 0.000704.
26. 1.01.
28. 1000.
29. 99.9.

Exercise 46

1. 1.8341.
2. 0.8938.
4. 9.9944 − 10.
5. 8.9941 − 10.
7. 4.8066.
8. 1.9960.
10. 7.0484 − 10.
11. 8.0004 − 10.
13. 4.9566.
14. 1.0216.
16. 9.9087 − 10.
17. 6.4881 − 10.
19. 3.2407.
20. 3.8820.
22. 4.9952.
23. 5.8724.
25. 594.7.
26. 6.463.
28. 1057.
29. 99650.
31. 0.004593.
32. 0.03011.
34. 0.1360.
35. 0.01490.
37. 1.710.
38. 17.70.
40. 2090.
41. 21600.
43. 0.5464.
44. 0.0492.
46. 0.01023.
47. 0.001084.

Exercise 47

1. 233.8.
2. 4.293.
4. 1.657.
5. 4.487.
7. 0.3555.
8. 95.28.
10. 0.5510.
11. 3.260.
13. 0.9670.
14. 68.52.
16. 0.1001.
17. 46.62.
19. 172.2.
20. 6173.
22. 85.68.
23. 292.5.
25. 0.4627.
26. 2.532.
28. 0.003805.
29. 0.00004126.
31. 7.390.
32. 36.51.
34. 0.6593.
35. 0.9445.
37. 82.72.
38. 22.77.
40. 4.230.
41. 2.007.
43. 0.4637.
44. 2.054.
46. 7.632.
47. 0.2735.
49. 0.04048.
50. 0.002402.
52. 0.8004.
53. 0.8124.
55. 0.2683.
56. 1.863.
58. 6.081.
59. 0.004640.
61. 4.250.
62. 50.51.

Exercise 48

1. 3.
2. 3.
4. 2.
5. 2.
7. −1.
8. −2.
10. 1.404.
11. 1.233.
13. 0.8857.
14. 1.887.
16. 0.
17. 0.8414.
19. 0.
20. 0.5756.
22. −0.2043.
23. −1.857.
25. 18.80.
26. −5.795.

Exercise 49

1. 4.605.
2. 5.063.
4. 5.755.
5. 4.701.
7. -1.265.
8. -1.679.
10. 2.090.
11. -3.194.
13. 1.100.
14. -2.144.

Exercise 50

1. ± 3.
2. $\pm\frac{9}{2}$.
4. ± 2.
5. ± 3.
7. $\pm\sqrt{3}$.
8. $\pm a$.
10. $\pm\dfrac{bi}{a}\cdot$
11. $\pm 3i$.
13. $\pm\dfrac{7i}{2}\cdot$
14. $\pm i\sqrt{7}$.
16. $\pm\dfrac{i}{2}\cdot$
17. $\pm\frac{i}{2}\sqrt{6}$.
19. $-1, 2$.
20. $-3, 2$.
22. 3, 4.
23. $-8, -1$.
25. $-2, \frac{5}{2}$.
26. $-\frac{3}{2}, -\frac{1}{3}$.
28. $-\frac{1}{2}, -\frac{1}{4}$.
29. $-\frac{2}{5}, 3$.
31. $-\frac{4}{5}, \frac{3}{2}$.
32. $\frac{2}{7}, 1$.
34. $-\frac{2}{3}, \frac{3}{2}$.
35. $\frac{3}{4}, 3$.
37. $a, 3a$.
38. $-\dfrac{2}{a}, \dfrac{1}{a}\cdot$
40. -2.
41. 3.
43. $-b, -1$.
44. a, b.
46. $1, \dfrac{b}{a}\cdot$
47. $-\dfrac{a}{b}, \dfrac{b}{a}\cdot$

Exercise 51

1. $-3, 1$.
2. 2, 3.
4. $-5, 4$.
5. $-4, -3$.
7. $-1, \frac{3}{2}$.
8. $\frac{1}{2}, 2$.
10. $\frac{1}{4}, \frac{1}{2}$.
11. $-\frac{2}{3}, \frac{3}{2}$.
13. $1 \pm \sqrt{2}$.
14. $-2 \pm \sqrt{3}$.
16. $3 \pm 2\sqrt{3}$.
17. $\dfrac{-2 \pm \sqrt{2}}{2}$.
19. $\dfrac{4 \pm \sqrt{5}}{3}$.
20. $\dfrac{7 \pm \sqrt{7}}{2}$.
22. $\dfrac{1 \pm 3\sqrt{2}}{5}$.
23. $\dfrac{3 \pm 2\sqrt{2}}{4}$.
25. $-3 \pm 2i$.
26. $4 \pm 3i$.
28. $\dfrac{3 \pm i\sqrt{3}}{2}$.
29. $\dfrac{2 \pm i\sqrt{6}}{3}$.
31. $2a, -\dfrac{a}{2}\cdot$
32. $-\dfrac{1}{a}, \dfrac{2}{a}\cdot$
34. $a - b, a + b$.
35. $b, a - b$.
37. $-2a, 3b$.

Exercise 52

1. $-3, 2$.
2. $-3, 4$.
4. $-5, -4$.
5. 5.
7. $-\frac{1}{2}, 2$.
8. $-1, \frac{2}{3}$.
10. $-\frac{5}{3}, 1$.
11. $-\frac{1}{4}, -\frac{1}{5}$.
13. $\dfrac{-1 \pm \sqrt{5}}{2}$.

14. $\dfrac{1 \pm \sqrt{17}}{2}$.
16. $\dfrac{-5 \pm \sqrt{17}}{4}$.
17. $\dfrac{-5 \pm \sqrt{13}}{3}$.
19. $\dfrac{9 \pm \sqrt{33}}{8}$.
20. $\dfrac{-11 \pm \sqrt{37}}{6}$.
22. $\dfrac{1 \pm i\sqrt{3}}{2}$.
23. $1 \pm i$.
25. $\dfrac{1 \pm 3i}{2}$.
26. $\dfrac{-5 \pm \sqrt{13}}{6}$.
28. $\dfrac{1 \pm i\sqrt{95}}{12}$.
29. $3a,\ -2a$.
31. $a,\ -b$.
32. $-1,\ -\dfrac{1}{m}$.
34. $\dfrac{3}{b},\ -\dfrac{2}{a}$.
35. $\dfrac{2}{a+b},\ -\dfrac{1}{a+b}$.
37. $-2a - b,\ -3a + b$.
38. $m - n,\ -2(m - n)$.
40. $x = 2y,\ x = 2 - y;\ y = \frac{1}{2}x,\ y = 2 - x$.
41. $x = y + 1,\ x = \frac{3}{2}y;\ y = x - 1,\ y = \frac{2}{3}x$.
43. $x = -3y,\ x = \dfrac{y-2}{3};\ y = -\frac{1}{3}x,\ y = 3x + 2$.
44. $x = 1 - y,\ x = -y - 2;\ y = 1 - x,\ y = -x - 2$.
46. $x = 2y - 1 \pm \sqrt{y};\ y = \dfrac{4x + 5 \pm \sqrt{8x + 9}}{8}$.

Exercise 53

1. $\pm 1,\ \pm 3$.
2. $\pm\sqrt{2},\ \pm i$.
4. $\pm\frac{1}{2}\sqrt{2},\ \pm 2i$.
5. $\pm 2i,\ \pm i\sqrt{3}$.
7. $-\frac{1}{4},\ \frac{1}{3}$.
8. $-1,\ 2$.
10. $-\frac{4}{5},\ 1$.
11. $\pm 1,\ \pm\frac{1}{4}$.
13. $-64,\ -27$.
14. $\pm 1,\ \pm 8$.
16. 25.
17. 16.
19. 1.
20. $\frac{1}{9},\ \frac{1}{4}$.
22. $-1 \pm \sqrt{5}$.
23. $\pm 3,\ \pm i$.
25. $-2,\ -1,\ 2,\ 3$.
26. $1,\ 4,\ \dfrac{5 \pm \sqrt{37}}{2}$.
28. $-\frac{5}{3},\ -1,\ \frac{1}{3},\ 1$.
29. $-1,\ \frac{1}{2}$.
31. $-\frac{19}{3},\ 3$.
32. $-3,\ -\frac{9}{4}$.
34. $-\frac{1}{4},\ 1,\ \dfrac{3 \pm \sqrt{89}}{8}$.
35. $-1,\ 2,\ -1 \pm \sqrt{6}$.

Exercise 54

1. 5.
2. 61.
4. ± 2.
5. $-1,\ 4$.
7. $-\frac{1}{2}$.
8. No solution.
10. $\frac{1}{7}$.
11. $-1,\ 2$.
13. 7.
14. -3.
16. 6.
17. 6.
19. 7.
20. 3.
22. 4.
23. 0, 2.
25. 2.
26. ± 1.
28. 3, 4.
29. -2.
31. $3,\ -\frac{4}{3}$.

Exercise 55

1. Real, equal, rational.
2. Real, unequal, rational.
4. Imaginary, unequal.
5. Imaginary, unequal.
7. Real, unequal, irrational.
8. Imaginary, unequal.
10. Real, equal, irrational.
11. Real, unequal, irrational.
13. Real, equal, rational.
14. Real, unequal, rational.

16. $\frac{9}{4}$.
17. 3.
19. 1, 13.
20. 2, $-\frac{13}{2}$.
22. -1, $-\frac{1}{5}$.

Exercise 56

1. 2, $\frac{3}{4}$.
2. $\frac{2}{3}$, -2.
4. -5, -7.
5. 3, $-\frac{3}{2}$.
7. $-\frac{7}{4}$, -1.
8. 0, 9.
10. -3, 0.
11. 0, $\frac{3}{5}$.
13. $-\frac{7}{2}$, $-\frac{9}{2}$.
14. $\frac{5}{8}$, $\frac{3}{4}$.
16. $x^2 - x - 2 = 0$.
17. $x^2 - 7x + 12 = 0$.
19. $9x^2 + 6x - 8 = 0$.
20. $10x^2 - 29x + 10 = 0$.
22. $x^2 - 2 = 0$.
23. $x^2 + 4 = 0$.
25. $x^2 - 4x + 2 = 0$.
26. $x^2 - 2x - 2 = 0$.
28. $9x^2 - 12x + 5 = 0$.
29. $25x^2 - 20x - 14 = 0$.
31. 12.
32. 3.
34. 21.
35. 0.
37. ± 9.
38. $\frac{27}{64}$.
40. 10.
41. 2.
43. -2.
44. 4.

Exercise 57

16. 2, min -6.
17. 2, max 10.
19. 2, max 20.
20. 1, min 0.
22. 0, min $\frac{7}{8}$.
23. 0, max $-\frac{9}{2}$.
25. 0, min $\frac{13}{4}$.
26. 1, min 0.
28. 12,100 sq rd.
29. 9 and 9.
31. 4 and 4.

Exercise 58

1. 11, 13 and -13, -11.
2. 6, 7 and -7, -6.
4. 5, 6 and -6, -5.
5. 8 in. by 11 in.
7. 20 in. by 21 in.
8. 1 in.

10. Depth $4\frac{1}{2}$ in., width 9 in.
11. 15 ft.
13. 9.06 in. by 15.06 in.
14. 40 and 55 mph.

16. Approx. 57 minutes after the second ship starts.
17. 1 hr and 1.56 hr after the first man is at the intersection.

19. 150 mph.
20. 12 min and 20 min.
22. 80.
23. 40.
25. 21 days and 28 days.
26. 2 sec.
28. 10 sec.

Exercise 60

1. $(-4, \pm 3)$, $(4, \pm 3)$.
2. $(-2, \pm 5)$, $(2, \pm 5)$.
4. $(-\sqrt{7}, \pm\sqrt{3})$, $(\sqrt{7}, \pm\sqrt{3})$.
5. $(-\sqrt{11}, \pm\sqrt{5})$, $(\sqrt{11}, \pm\sqrt{5})$.
7. $(-i, \pm 2i)$, $(i, \pm 2i)$.
8. $(-4i, \pm 3i)$, $(4i, \pm 3i)$.
10. $(1, \pm i\sqrt{2})$, $(-1, \pm i\sqrt{2})$.
11. $(-i\sqrt{3}, \pm\sqrt{5})$, $(i\sqrt{3}, \pm\sqrt{5})$.

Exercise 61

1. $(2, -1), (-4, 11)$.
2. $(-1, 3), (-21, -7)$.
4. $(1, 3), (3, 1)$.
5. $(-2, 3), (3, -2)$.
7. $(3, -3), (-1, 5)$.
8. $(2 + i, 3 - i), (2 - i, 3 + i)$.
10. $(2, 1), (-4, -3)$.
11. $(4, 3), (-2, -5)$.
13. $(3, 0), (5, -2)$.
14. $(4, 1), (6, 4)$.
16. $(-1, 1), (-\frac{1}{9}, \frac{1}{3})$.
17. $\left(\frac{-3\sqrt{5}}{4}, \frac{-3 - 3\sqrt{5}}{2}\right), \left(\frac{3\sqrt{5}}{4}, \frac{-3 + 3\sqrt{5}}{2}\right)$.

Exercise 62

1. $(2, 2), (-2, -2), (1, 2), (-1, -2)$.
2. $(1, 2), (-1, -2), (\sqrt{5}, -2\sqrt{5}), (-\sqrt{5}, 2\sqrt{5})$.
4. $(-2, 2), (2, -2)$.
5. $(0, \pm i\sqrt{2}), (3, 1), (-3, -1)$.
7. $(-2, 3), (2, -3), (-\frac{1}{3}, \frac{5}{3}), (\frac{1}{3}, -\frac{5}{3})$.
8. $(\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}), \left(\frac{5\sqrt{87}}{29}, -\frac{\sqrt{87}}{29}\right), \left(-\frac{5\sqrt{87}}{29}, \frac{\sqrt{87}}{29}\right)$.
10. $(3, -2), (-3, 2), \left(\frac{8\sqrt{7}}{7}, -\frac{4\sqrt{7}}{7}\right), \left(-\frac{8\sqrt{7}}{7}, \frac{4\sqrt{7}}{7}\right)$.
11. $(i, 5i), (-1, -5i), \left(\frac{5\sqrt{119}}{17}, \frac{\sqrt{119}}{17}\right), \left(-\frac{5\sqrt{119}}{17}, -\frac{\sqrt{119}}{17}\right)$.
13. $(2, 3), (-2, -3), \left(\frac{22}{\sqrt{158}}, -\frac{4}{\sqrt{158}}\right), \left(-\frac{22}{\sqrt{158}}, \frac{4}{\sqrt{158}}\right)$.
14. $(\sqrt{2}, -3\sqrt{2}), (-\sqrt{2}, 3\sqrt{2})$.
16. $(3, -1), (-3, 1), \left(\frac{i\sqrt{34}}{2}, \frac{3i\sqrt{34}}{2}\right), \left(-\frac{i\sqrt{34}}{2}, \frac{-3i\sqrt{34}}{2}\right)$.

Exercise 63

1. $(2, -3), (\frac{7}{2}, 0)$.
2. $(3, 1), (\frac{2}{3}, 3)$.
4. $(-3, -2), (\frac{4}{5}, \frac{62}{45})$.
5. $(\pm 2, -2), (\pm\frac{1}{2}\sqrt{41}, -\frac{9}{2})$.
7. $(3, 4), (-1, -2)$.
8. $(0, 1), (2, -3)$.
10. $(4, -1), (1, -\frac{5}{2})$.
11. $(2, 0), (1, \frac{1}{3})$.
13. $(3, 1), (1, 3), (-3 + 2i, -3 - 2i), (-3 - 2i, -3 + 2i)$.
14. $(2, 2), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.
16. $(4, 2), (2, 4), (8, 6), (6, 8)$.

Exercise 64

1. 7 and 9.
2. 8 by 12.
4. 10, 13, 13.
5. 4 in., 4 in., 2 in.
7. 9 in., 16 in.
8. 60¢, 40.
10. 3%, $2800.
11. 12 hr and 24 hr.
13. 200 mph, 800 mi.

Exercise 65

1. 16, 19.
2. 0, -2.
4. 16, 20.
5. Not arithmetic.
7. Not arithmetic.
8. Not arithmetic.
10. $-\frac{2}{3}$, -2.
11. 1.7, 1.
13. $8x - 11y$, $10x - 13y$.
14. $-x + 16y$, $-3x + 26y$.
16. -28.
17. 2.
19. -123.
20. 25.3.
22. $23\frac{1}{12}$.
23. -47.2.
25. $13x - 12y$.
26. $d = \frac{9}{4}$.
28. $d = \frac{9}{2}$.
29. $d = -1$.
31. $d = \frac{7}{2}$.

Exercise 66

1. 70, 648.
2. 45, 529.
4. -52, -255.
5. 12, 0.
7. $\frac{7}{4}$, 10.
8. 31, $300\frac{1}{3}$.
10. $n = 14$, $S = 248\frac{1}{2}$.
11. $a = 4$, $S = 220$.
13. $a = -25$, $d = 5$.
14. $n = 13$, $d = \frac{3}{2}$.
16. $n = 4$, $a = -9$; $n = 12$, $a = 7$.
17. $n = 12$, $l = -21$.
19. $n = 8$, $a = -13$.
20. $n = 6$, $a = 8$; $n = 9$, $a = -4$.
22. 1170.
23. 990.
25. 406.
26. \$32.40, $n = 141$.
28. 15 sec.
29. \$56,400.
31. \$465.
32. The second proposition pays \$50 more per year than the first.
35. $a = \dfrac{S}{n} - \dfrac{(n-1)d}{2}$, $l = \dfrac{S}{n} + \dfrac{(n-1)d}{2}$.

Exercise 67

1. 192, 381.
2. $\frac{9}{64}$, $35\frac{55}{64}$.
4. $\frac{128}{81}$, $50\frac{68}{81}$.
5. $\frac{6561}{1024}$, $172\frac{797}{1024}$.
7. $5\frac{61}{64}$.
8. -42.
10. $20\frac{7}{27}$.
11. $-20\frac{5}{8}$.
13. $r = 2$.
14. $r = \frac{1}{3}$.
16. $r = \frac{2}{3}$.
17. $r = \frac{3}{5}$.
19. $a = \frac{1}{3}$, $S = 121\frac{1}{3}$.
20. $n = 5$, $S = 26\frac{8}{9}$.
22. $n = 3$, $l = 2\frac{2}{5}$.
23. $r = \frac{1}{4}$, $l = 1$.
25. $r = \frac{1}{5}$, $a = \frac{50}{3}$; $r = -\frac{1}{6}$, $a = 24$.
26. $a = 27$, $n = 4$.
28. $r = -2$, $a = 6$.
29. \$592.59.
31. 3,543,122.
32. 59%, 23.
34. \$231.85.
35. $126\frac{8}{27}$.
37. \$42,949,672.
38. 40.95%.

40. The first proposition pays \$5800 for the fifth year and \$24,500 for the five-year period. The second proposition pays \$5856.40 for the fifth year and \$24,420.40 for the five-year period.

Exercise 68

1. $\frac{3}{2}$.
2. $\frac{1}{3}$.
4. $7\frac{1}{9}$.
5. $12\frac{1}{2}$.
7. $\frac{5}{9}$.
8. 0.2.
10. $6\frac{2}{3}$.
11. $-30\frac{6}{7}$.
13. $\frac{8}{9}$.
14. $\frac{5}{9}$.
16. $\frac{34}{99}$.
17. $\frac{7}{33}$.

19. $\frac{103}{33}$.
20. $\frac{281}{99}$.
22. $\frac{41}{333}$.
23. $\frac{21}{37}$.
25. $\frac{1}{7}$.
26. $\frac{37}{481}$.
28. 2, $1\frac{511}{512}$.
29. 0.9, 0.89986.
31. 600 ft.
32. 100 in.

Exercise 69

13. $x < 4$.
14. $x > 4$.
16. $x < -\frac{1}{2}$.
17. $x < -\frac{1}{3}$.
19. $x < \frac{9}{2}$.
20. $x < \frac{5}{2}$.
22. $-5 < x < 5$.
23. $x < -4$ and $x > 4$.
25. $1 < x < 3$.

26. $-2 < x < 3$.
28. No solution.
29. All real values except $x = 0$.
31. All real values.

32. All real values of x.
34. $-\frac{2}{3} < x < 2$.
35. $1 < x < \frac{7}{2}$.

37. $x < 3$ but not equal to 2.
38. $x < 5$ but not equal to -2.

40. $x < -3,\ -1 < x < 2$.
41. $-5 < x < 3,\ x > 6$.
43. $-2 < x < 10$.
44. $x < -4,\ x > 4$.
46. $-2 < x < 8$.
47. $x < 1,\ x > 3$.
49. $-3 < x < 4$.
50. $x < \frac{1}{3},\ x > \frac{7}{3}$.
52. $1 < x < 2$.
53. $x < -5,\ x > 3$.
55. $x < -3,\ x > 0$.
56. $0 < x < 3$.

Exercise 70

1. 720.
2. 5040.
4. 48.
5. 144.
7. 702.
8. 230.
10. 8.
11. 90.
13. $\frac{1}{336}$.
14. 252.
16. n.
17. $n + 1$.
19. $n!$.
20. $n!$.

Exercise 71

1. $a^4 - 4a^3x + 6a^2x^2 - 4ax^3 + x^4$.
2. $a^4 + 8a^3y + 24a^2y^2 + 32ay^3 + 16y^4$.

4. $a^5 + 10a^4 + 40a^3 + 80a^2 + 80a + 32$.
5. $81a^4 + 216a^3y + 216a^2y^2 + 96ay^3 + 16y^4$.
7. $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$.
8. $1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8$.
10. $m^9 + 9m^8n + 36m^7n^2 + 84m^6n^3 + 126m^5n^4 + 126m^4n^5 + 84m^3n^6 + 36m^2n^7 + 9mn^8 + n^9$.
11. $m^{10} - 10m^9n + 45m^8n^2 - 120m^7n^3 + 210m^6n^4 - 252m^5n^5 + 210m^4n^6 - 120m^3n^7 + 45m^2n^8 - 10mn^9 + n^{10}$.
13. $64a^6 - 192a^5y + 240a^4y^2 - 160a^3y^3 + 60a^2y^4 - 12ay^5 + y^6$.
14. $32x^{10} - 80x^8y + 80x^6y^2 - 40x^4y^3 + 10x^2y^4 - y^5$.
16. $128x^7 - 448x^6y^2 + 672x^5y^4 - 560x^4y^6 + 280x^3y^8 - 84x^2y^{10} + 14xy^{12} - y^{14}$.
17. $a^{12} - 18a^{10}y + 135a^8y^2 - 540a^6y^3 + 1215a^4y^4 - 1458a^2y^5 + 729y^6$.
19. $\frac{x^6}{64} - \frac{3x^4}{8} + \frac{15x^2}{4} - 20 + \frac{60}{x^2} - \frac{96}{x^4} + \frac{64}{x^6}$.
20. $\frac{a^6}{b^6} + \frac{6a^4}{b^4} + \frac{15a^2}{b^2} + 20 + \frac{15b^2}{a^2} + \frac{6b^4}{a^4} + \frac{b^6}{a^6}$.
22. $x^3 - 3x^2y + 3xy^2 - y^3 + 3x^2z - 6xyz + 3y^2z + 3xz^2 - 3yz^2 + z^3$.
23. $8a^3 + 12a^2b + 6ab^2 + b^3 - 36a^2c - 36abc - 9b^2c + 54ac^2 + 27bc^2 - 27c^3$.

Exercise 72

1. $462a^5x^6$. **2.** $-792a^7x^5$. **4.** $1120x^4y^4$.

5. $495a^8b^8$. **7.** $70a^4b^4$. **8.** $\frac{73{,}216}{125}x^{18}y^4$.

10. $\frac{55}{729}$. **11.** $\frac{21}{2}$. **13.** 3003.

14. $\frac{5005}{64}$. **16.** $1 + x + x^2 + x^3 + \cdots$

17. $a^{-2} - 2a^{-3}x + 3a^{-4}x^2 - 4a^{-5}x^3 + \cdots$.

19. $a^{-4} - 12a^{-5} + 90a^{-6} - 540a^{-7} + \cdots$.

20. $\frac{1}{4^5} + \frac{5}{4^6}x + \frac{15}{4^7}x^2 + \frac{35}{4^8}x^3 + \cdots$.

22. $1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + \cdots$.

23. $3^{2/3}\left[1 + \frac{2y}{9} - \frac{y^2}{81} + \frac{4y^3}{2187} - \cdots\right]$.

25. $5^{-11/3}\left[125 - \frac{50}{3}b + \frac{25}{9}b^2 - \frac{40}{81}b^3 + \cdots\right]$.

26. $1 + \frac{9}{4}x + \frac{189}{32}x^2 + \frac{2079}{128}x^3 + \cdots$.

28. 3.037. **29.** 3.009. **31.** 0.990.

32. 1.025. **34.** 0.329. **35.** 0.497.

Supplementary Problems

1. $x^{-3} + 6x^{-2} + 12x^{-1} + 8$. **2.** $x^{3/2} + 3xy^{1/2} + 3x^{1/2}y + y^{3/2}$.

4. $x^5y^{10} - 10x^4y^8z^3 + 40x^3y^6z^6 - 80x^2y^4z^9 + 80xy^2z^{12} - 32z^{15}$.

5. $1 + 15x^{1/2} + 90x + 270x^{3/2} + 405x^2 + 243x^{5/2}$.

7. $x^3 - 3x^{5/2}y + \frac{15}{4}x^2y^2 - \frac{5}{2}x^{3/2}y^2 + \frac{15}{16}xy^4 - \frac{3}{16}x^{1/2}y^5 + \frac{1}{64}y^6$.

8. $a^{5/2} - 5a^2b^{1/2} + 10a^{3/2}b - 10ab^{3/2} + 5a^{1/2}b^2 - b^{5/2}$.

11. 1.105. **13.** 2.5. **14.** 0.86.

16. 9604. **17.** 970,299. **19.** 3025.

20. 8464.

Exercise 74

1. 0. **2.** 3. **4.** -9.

5. -25. **7.** -50. **8.** -193.

10. $-\frac{22}{9}$. **11.** 47. **13.** 43.

14. -5. **32.** $x^3 - 7x + 6$.

34. $x^3 + 6x^2 + 11x + 6 = 0$. **35.** $x^3 - 3x^2 - 4x + 12 = 0$.

37. $x^3 + x^2 - 12x = 0$. **38.** $2x^3 - 7x^2 + 7x - 2 = 0$.

40. $9x^3 - 7x + 2 = 0$. **41.** $x^3 - 2x^2 - 2x + 4 = 0$.

43. $x^3 - 3x = 0$.

Exercise 75

1. $2x^2 - 5x - 2, -1$. **2.** $x^2 + 4x - 9, 3$. **4.** $3x^2 - 3x + 11, -21$.

5. $4x^2 - 8x + 25, -70$. **7.** $5x^2 + 3x + 4, 0$. **8.** $4x^2 + 3, -3$.

10. $x^3 + x^2 + 2x - 3, -8.$ **11.** $2x^3 + x^2 - 3x + 10, -17.$
13. $x^4 + x^3 + 2x^2 + 3x + 6, 13.$ **14.** $x^4 - 2x^3 + 5x^2 - 10x + 21, -41.$
16. $x^4 + x^3 - x^2 - 6, -23.$ **17.** $2x^4 - 2x^3 - 2x^2 + 4, 6.$

19. $-19, -19, -46.$ **20.** $54, 152, 14.$ **22.** $-25, 2, 57.$

Exercise 76

1. $-0.9, 1.7, 3.2.$ **2.** $0.8, -1.5, -3.2.$ **4.** $-2.2, -0.8, 2.9.$
5. $-3.1, -0.8, 2.9.$ **7.** $-1.7, -0.2, 2.9.$ **8.** $-4.9, 1.5, 3.4.$
10. $-3.1.$ **11.** $3.2.$ **13.** $-1.7, 1.7.$
14. $-1.7, -1, 1, 1.7.$ **16.** $0.3, 1.5.$

Exercise 77

1. $-i, -4.$ **2.** $2 + 3i, -1.$ **4.** $-\sqrt{5}, -\frac{3}{2}.$
5. $1 + \sqrt{2}, \frac{1}{3}.$ **7.** $1 + 2i, \pm 1.$ **8.** $-\sqrt{7}, 2 \pm 2i.$

10. $-2 + \sqrt{3}, \pm i.$ **11.** $x^3 - 2x + 4 = 0.$
13. $x^4 - 4x^3 + 6x^2 - 4x + 5 = 0.$ **14.** $x^4 - 10x^3 + 27x^2 + 20x - 58.$
16. $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0.$ **17.** $x^4 - 6x^3 - 3x^2 + 42x - 28 = 0.$
19. $x^4 - 12x^3 + 65x^2 - 212x + 260 = 0.$

20. $-4, -7.$ **22.** $0, 1.$ **23.** $7, 0.$

25. $0, 3.$ **26.** $\frac{5}{2}, -4.$
28. $x^3 - 3x^2 - 6x + 8 = 0.$ **29.** $x^3 + 2x^2 - 15x = 0.$
31. $x^3 - 7x^2 + 14x - 8 = 0.$ **32.** $x^3 + 3x^2 - 4 = 0.$
34. $x^3 + 2x^2 - 4x - 8 = 0.$ **35.** $x^4 - 2x^3 - x^2 + 2x = 0.$

Exercise 78

1. $-3, 1.$ **2.** $-1, 3.$ **4.** $-2, 1.$
5. $-1, 4.$ **7.** $-3, 4.$ **8.** $-3, 3.$
10. $-3, 3.$ **11.** $-6, 5.$ **13.** $-2, 2.$
14. $-4, 3.$ **16.** $-1, 2.$ **17.** $-2, 3.$

19. One positive root and two imaginary roots.
20. One negative root and two imaginary roots.
22. One negative root and two imaginary roots.
23. One negative root. The other two are either positive or imaginary.
25. Three roots are positive or one root is positive and the others imaginary.
26. Three roots are positive or one root is positive and the others imaginary.
28. One positive root, one negative root, and two imaginary roots.

29. There is one positive root. The other roots are all negative or one is negative and two imaginary.

31. There is no positive root. There are four, two, or no negative roots. Hence there are zero, two, or four imaginary roots.

32. There are two positive roots or no positive roots, and two negative roots or no negative roots. Hence there are zero, two, or four imaginary roots.

34. There are either three positive roots or one positive root. There is one negative root. Hence there are two or four imaginary roots.

Exercise 79

1. $1, \pm 2i.$
2. $-1, \pm i\sqrt{3}.$
4. $-2, -2, 3.$
5. $1, 1, -4.$
7. $3, 1 \pm i.$
8. $2, 2 \pm 3i.$
10. $\frac{1}{3}, \pm i.$
11. $\frac{1}{3}, \pm 3i.$
13. $-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}.$
14. $-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}.$
16. $-\frac{2}{3}, -1 \pm 2i.$
17. $2.$
19. $-3, -2, \pm i.$
20. $-2, 3, \dfrac{-1 \pm \sqrt{5}}{2}.$
22. $0, -1, 2, 4.$
23. $-\frac{1}{3}, 0, \pm i.$
25. $-\frac{1}{2}, \frac{1}{3}, \pm i.$
26. $-\frac{3}{2}, \frac{2}{3}, \pm i.$
28. $(x - 2)(4x^2 + 4x + 5).$
29. $(2x + 1)(x^2 - 4x + 11).$
31. $(x - 1)(x + 1)(x + 2)^2.$
32. $(x - 2)(x + 2)(x^2 - 4x + 5).$
34. $(x + 1)^2(x - 1)(x + 2)(x - 2).$

Exercise 80

1. 1.51.
2. 2.27.
4. 1.68.
5. 2.04.
7. 0.33.
8. 2.93.
10. 0.36.
11. 1.38.
13. −1.11, 1.25, 2.86.
14. −0.83, 0.34, 3.49.
16. −0.94, 2.46, 3.47.
17. −0.25, 1.45, 2.80.
19. −0.47, 1.12, 2.35.
20. −0.93, 0.52, 2.08.
22. 0.89.
23. 2.07.

Exercise 81

1. $x^2 + 3x + 5 = 0.$
2. $3x^2 - 6x - 1 = 0.$
4. $2x^3 - 3x - 4 = 0.$
5. $x^4 - 6x^3 - 4x^2 + 2x - 1 = 0.$
7. $x^2 - 3x + 2 = 0.$
8. $x^2 - 5x + 6 = 0.$
10. $x^3 + x^2 - x - 6 = 0.$
11. $3x^3 + 4x^2 - x - 14 = 0.$
13. $4x^3 - 4x^2 + x + 2 = 0.$
14. $2x^3 - 11x^2 + 12x = 0.$
16. $x^3 - 4.4x^2 + 1.12x - 1.592 = 0.$
17. $x^3 - 8.5x^2 - 9.25x - 3.375 = 0.$

Exercise 82

1. 0.513.
2. 1.272.
4. 3.691.
5. 2.328.
7. 2.376.
8. 1.550.
10. 2.382.
11. −2.115, 0.254, 1.861.
13 −1.935, 1.463, 2.473.
14. −1.247, 0.445, 1.802.
16. −1.929, −0.482, 1.077.
17. −1.473, 0.126, 1.347.
19. −0.93, 2.42.
20. 0.36, 3.06.
22. −1.58, 0.88.
23. −0.90, 1.50.

Exercise 83

22. $\sqrt{2}\,(\cos 45^\circ + i \sin 45^\circ).$
23. $4\sqrt{2}\,(\cos 315^\circ + i \sin 315^\circ).$
25. $2(\cos 60^\circ + i \sin 60^\circ).$
26. $4(\cos 300^\circ + i \sin 300^\circ).$
28. $2(\cos 90^\circ + i \sin 90^\circ).$
29. $3(\cos 180^\circ + i \sin 180^\circ).$
31. $2(\cos 45^\circ + i \sin 45^\circ).$
32. $16(\cos 330^\circ + i \sin 330^\circ).$

34. $2\sqrt{7}\,(\cos 300° + i \sin 300°)$.
35. $5(\cos 53° + i \sin 53°)$.
37. $\sqrt{29}\,(\cos 338° + i \sin 338°)$.
38. $25(\cos 254° + i \sin 254°)$.

40. $\sqrt{3} + i$.
41. $4 + 4i\sqrt{3}$.
43. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

44. $-\frac{5\sqrt{3}}{2} + \frac{5}{2}i$.
46. $-6i$.
47. $20i$.
49. $1.26 + 1.55i$.

Exercise 84

1. $12i$.
2. -12.
4. $-\frac{7}{2} + \frac{7\sqrt{3}}{2}i$.

5. $-\frac{3\sqrt{3}}{2} + \frac{3}{2}i$.
7. $2i$.
8. -9.

10. $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$.
11. 12.
13. $0.643 + 0.766i$.

14. $0.766 + 0.643i$.
16. $2\sqrt{2}\,(\cos 75° + i \sin 75°)$.
17. $6\sqrt{2}\,(\cos 345° + i \sin 345°)$.
19. $8i$.
20. $20\sqrt{2}\,(\cos 135° + i \sin 135°)$.
22. $8\sqrt{2}\,(\cos 315° + i \sin 315°)$.
23. $\frac{\sqrt{2}}{2}(\cos 345° + i \sin 345°)$.
25. $\frac{3\sqrt{2}}{4}(\cos 45° + i \sin 45°)$.
26. $\frac{2}{3}(\cos 30° + i \sin 30°)$.
28. $4(\cos 330° + i \sin 330°)$.

Exercise 85

1. $9(\cos 40° + i \sin 40°)$.
2. $8(\cos 105° + i \sin 105°)$.
4. $25(\cos 80° + i \sin 80°)$.
5. $256(\cos 200° + i \sin 200°)$.
7. $\cos 340° + i \sin 340°$.
8. $128(\cos 300° + i \sin 300°)$.
10. $8(\cos 90° + i \sin 90°)$.
11. $16(\cos 240° + i \sin 240°)$.
13. $1024(\cos 60° + i \sin 60°)$.
14. $128\sqrt{2}\,(\cos 135° + i \sin 135°)$.

16. $5(\cos 67° + i \sin 67°)$, $5(\cos 247° + i \sin 247°)$.
17. $3(\cos 50° + i \sin 50°)$, $3(\cos 170° + i \sin 170°)$, $3(\cos 290° + i \sin 290°)$.
19. $\cos 20° + i \sin 20°$, $\cos 92° + i \sin 92°$, $\cos 164° + i \sin 164°$, $\cos 236° + i \sin 236°$, $\cos 308° + i \sin 308°$.
20. $2(\cos 70° + i \sin 70°)$, $2(\cos 160° + i \sin 160°)$, $2(\cos 250° + i \sin 250°)$, $2(\cos 340° + i \sin 340°)$.
22. $\cos 30° + i \sin 30°$, $\cos 150° + i \sin 150°$, $\cos 270° + i \sin 270°$.
23. $\cos 67\frac{1}{2}° + i \sin 67\frac{1}{2}°$, $\cos 157\frac{1}{2}° + i \sin 157\frac{1}{2}°$, $\cos 247\frac{1}{2}° + i \sin 247\frac{1}{2}°$, $\cos 337\frac{1}{2}° + i \sin 337\frac{1}{2}°$.
25. 3, $3(\cos 90° + i \sin 90°)$, $3(\cos 180° + i \sin 180°)$, $3(\cos 270° + i \sin 270°)$.
26. $3(\cos 60° + i \sin 60°)$, $3(\cos 180° + i \sin 180°)$, $3(\cos 300° + i \sin 300°)$.

28. -243.
29. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

31. i.
32. $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

34. $1.38 + 0.0547i$, $0.376 + 1.33i$, $-1.15 + 0.769i$, $-1.09 - 0.858i$, $0.480 - 1.30i$.
35. $-0.493 + 1.45i$, $-1.01 - 1.15i$, $1.50 - 0.299i$.

37. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

38. $-1, 1, \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}.$

40. $-1, i.$

41. $\frac{2-1\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \frac{2+\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

Exercise 86

1. $\frac{3}{x+1} - \frac{2}{x-1}.$

2. $\frac{1}{x} - \frac{1}{x+3}.$

4. $\frac{3}{2x-1} + \frac{2}{x-2}.$

5. $\frac{4}{3x+1} - \frac{3}{3x-1}.$

7. $\frac{2}{3x+1} + \frac{3}{x-1}.$

8. $\frac{1}{2x+3} + \frac{1}{x-2}.$

10. $\frac{2}{2x+3} + \frac{4}{2x-3} - \frac{3}{x}.$

11. $\frac{1}{x} + \frac{1}{x-4} - \frac{2}{x+2}.$

13. $\frac{5}{x-3} - \frac{4}{x-4} + \frac{3}{x+2}.$

14. $\frac{1}{x+2} + \frac{1}{x-2} - \frac{2}{x+1}.$

16. $\frac{-1}{1-x} + \frac{8}{2-x} - \frac{4}{3-x}.$

Exercise 87

1. $\frac{1}{x-4} - \frac{1}{(x-4)^2}.$

2. $\frac{3}{(2x-1)^2} - \frac{4}{2x-1}.$

4. $\frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{3}{(x-1)^3}.$

5. $\frac{2}{3x+1} - \frac{3}{(3x+1)^2} + \frac{5}{(3x+1)^3}.$

7. $\frac{3}{x-2} - \frac{1}{x-1} + \frac{1}{(x-1)^2}.$

8. $\frac{10}{x+6} - \frac{4}{x+1} - \frac{3}{(x+1)^2}.$

10. $\frac{2}{x} - \frac{3}{x^2} - \frac{2}{x-3} + \frac{14}{(x-3)^2}.$

11. $\frac{1}{x+1} - \frac{2}{(x-1)^3}.$

13. $\frac{1}{2x-3} - \frac{1}{3x+2} + \frac{1}{x-2} + \frac{3}{(x-2)^2}.$

14. $\frac{2}{3x+1} - \frac{1}{2x-1} + \frac{1}{x+3} + \frac{2}{(x+3)^2}.$

16. $\frac{1}{2x-1} + \frac{2}{(2x-1)^2} + \frac{1}{x-2} + \frac{1}{(x-2)^2}.$

Exercise 88

1. $\frac{1}{x-2} + \frac{3}{x^2+5}.$

2. $\frac{1}{x+1} - \frac{1}{x^2+3}.$

4. $\frac{-5}{x-2} - \frac{3}{2x^2+2x+1}.$

5. $\frac{4}{x-4} - \frac{3x+2}{x^2-x-5}.$

7. $\frac{1}{x+2} - \frac{1}{x+1} + \frac{3x}{x^2+2}.$

8. $\frac{1}{x} + \frac{3}{x-1} + \frac{2}{x^2+4}.$

10. $\frac{3x}{x^2+x+1} - \frac{2}{(x-1)^2}.$

11. $\frac{3x+2}{x^2+1} - \frac{2x-3}{x^2+2}.$

13. $\frac{x+1}{2x^2+x+2} - \frac{x}{3x^2-x+1}.$

14. $\frac{3x}{x^2+2x+2} + \frac{2}{x^2-2x+2}.$

Exercise 89

1. $\dfrac{3x}{x^2+2} - \dfrac{2x+5}{(x^2+2)^2}$.
2. $\dfrac{x+1}{x^2+1} - \dfrac{2x-3}{(x^2+1)^2}$.
4. $\dfrac{x+1}{(x^2+2)^2} + \dfrac{2x-1}{(x^2+2)^3}$.
5. $\dfrac{1}{x^2-x+4} - \dfrac{3x-5}{(x^2-x+4)^3}$.
7. $\dfrac{1}{x^2+1} + \dfrac{2}{(x^2+1)^2} - \dfrac{x}{(x^2+1)^3}$.
8. $\dfrac{3}{x} - \dfrac{1}{x^2-2x+3} - \dfrac{2x-3}{(x^2-2x+3)^2}$.
10. $\dfrac{1}{x^2} - \dfrac{2}{2x^2+1} + \dfrac{x-4}{(2x^2+1)^2}$.
11. $\dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{x}{(x^2+5)^2} - \dfrac{2}{x^2+5}$.
13. $\dfrac{x-3}{(x^2-2x+2)^2} + \dfrac{2x}{x^2-2x+2} - \dfrac{1}{x-3}$.
14. $\dfrac{2}{2x-1} + \dfrac{x+1}{x^2+x+2} + \dfrac{2x-5}{(x^2+x+2)^2}$.

Exercise 90

1. 16, 12.
2. 125, 60.
4. 375, 72.
5. 30,000.
7. 1024.
8. 30, 36.
10. 6,760,000; 3,276,000.
11. 210.
13. 144.
14. 72.
16. 2880; 17,280; 5760.
17. 72, 144.
19. 30,240; 17,280; 8640.
20. 75, 126.

Exercise 91

1. 7, 210, 840, 5040.
2. 64.
4. 120, 20, 60.
5. 5040; 40,320; 604,800.
7. 60.
8. 72.
10. 60,480.
11. 4320, 720.
13. 34,560.
14. 12, 630, 1512.
16. 1260.
17. 90,090.
19. 13,860.
20. 720, 5040.
22. 80,640.
23. 1440, 3600.

Exercise 92

1. 465; 2600; 10,626; 1287.
2. 1820.
4. 66.
5. 36, 84.
7. 35.
8. 330, 462, 672.
10. 112.
11. 1960.
13. 20,592.
14. 8820; 16,170.
16. 246,960.
17. 127.
19. 1023.
20. 42.
22. 70.
23. 90.
25. 210.

Exercise 93

1. $\frac{1}{2}, \frac{1}{6}$.
2. $\frac{1}{28}, \frac{1}{4}, \frac{1}{7}$.
4. $\frac{1}{3}, \frac{1}{2}$.
5. $\frac{1}{4}, \frac{1}{2}$.
7. $\frac{1}{18}, \frac{7}{18}, \frac{1}{18}$.
8. $\frac{1}{32}, \frac{5}{32}, \frac{5}{16}$.
10. $\frac{1}{120}, \frac{1}{12}$.
11. $\frac{1}{14}, \frac{1}{4}$.
13. $\frac{1}{221}, \frac{1}{17}, \frac{4}{17}$.
14. $\frac{1}{5}, \frac{2}{5}$.
16. $\frac{24}{95}, \frac{111}{1330}$.
17. $\frac{8}{63}$.
19. $\frac{11}{850}, \frac{22}{425}, \frac{169}{425}$.
20. $\frac{1}{64}, \frac{1}{16}, \frac{3}{8}$.

Exercise 94

1. $\frac{16}{81}, \frac{25}{81}, \frac{20}{81}$.
2. $\frac{1}{6}, \frac{5}{18}, \frac{5}{18}$.
4. $\frac{1}{2197}, \frac{27}{2197}, \frac{1}{64}$.
5. $\frac{1}{5525}, \frac{11}{1105}, \frac{11}{850}$.
7. $\frac{1}{16}, \frac{3}{8}$.
8. $\frac{21}{506}, \frac{63}{253}, \frac{175}{2024}$.
10. $\frac{19}{30}$.
11. $\frac{7}{44}, \frac{7}{66}, \frac{7}{44}$.
13. $\frac{1}{462}, \frac{1}{462}$.
14. $\frac{3}{4}$, \$375.
16. $\frac{3}{4}, \frac{3}{16}$.
17. \$66$\frac{2}{3}$, \$25, \$6$\frac{2}{3}$.
19. \$3.60, \$4.29.
20. \$7.07, \$8.40.

Exercise 95

1. $\frac{3}{8}, \frac{1}{4}, \frac{5}{16}$.
2. $\frac{3125}{7776}, \frac{625}{3888}, \frac{26}{7776}$.
4. $\frac{1}{64}, \frac{6}{64}, \frac{15}{64}, \frac{20}{64}, \frac{15}{64}, \frac{6}{64}, \frac{1}{64}$; 3.
5. 0.2401, 0.4116, 0.3483.
7. 2, 1.
8. 0.096, 0.189, 0.288.
10. $\frac{243}{1024}, \frac{405}{1024}, \frac{135}{512}, \frac{45}{512}, \frac{15}{1024}, \frac{1}{1024}$.
11. 0.887, 0.770, 0.602.
13. \$4,704.45.
14. \$843.14, \$914.15.
16. 0.702, 0.271.
17. 0.271, 0.509.
19. 0.697, 0.263.

Exercise 97

1. -96.
2. 618.
4. -132.
5. -26.
7. 406.
8. 4.
10. -225.
11. 1083.
13. 27.

Exercise 98

1. $x = \frac{17}{8}, y = -\frac{1}{8}$.
2. $x = 19, y = 25$.
4. $x = 1, y = 2, z = -3$.
5. $x = 3, y = 0, z = 2$.
7. $x = 1, y = -1, z = 1, w = -1$.
8. $x = 2, y = -3, z = -2, w = -1$.
10. $x = 2, y = 0, z = -3, w = -4$.
11. $x = \frac{7}{3}, y = \frac{2}{3}, z = 1, w = -1$.
13. $x = -2, y = 3, z = \frac{3}{2}, w = -\frac{1}{2}$.
14. $x = \frac{11}{5}, y = -\frac{4}{5}, z = \frac{1}{2}, w = -\frac{1}{2}$.

Exercise 99

1. $x = \frac{1}{2}, y = -\frac{1}{3}$.
2. Inconsistent.
4. $x = -4, y = 2$.
5. $x = 4, y = -3$.
7. Inconsistent.
8. $x = 2, y = 2, z = -2$.
10. $x = 1, y = 3, z = -2$.
11. Inconsistent.
13. $x = 0, y = 0$.
14. $x = \frac{2}{3}k, y = k$.
16. $x = 2k, y = -3k, z = k$.
17. $x = k, y = \frac{1}{3}k, z = k$.
19. $x = 2k, y = k, z = k, w = -4k$.
20. $x = \frac{1}{4}k, y = \frac{1}{2}k, z = \frac{3}{4}k, w = k$.

INDEX

Abscissa, 67
Absolute inequality, 197
Absolute value, 3
  of a complex number, 244
Addition, algebraic, 3
  associative law for, 3
  commutative law for, 3
  of complex numbers, 120, 242
  of fractions, 42
  of radicals, 114
Algebraic expression, 2
American Experience Mortality Table, 304
Antilogarithm, 131
Arithmetic means, 183
Arithmetic progression, 182
  common difference of an, 182
  elements of an, 185
  extremes of an, 183
  means of an, 183
  *n*th term of an, 183
  sum of an, 184
Associative law, for addition, 3
  for multiplication, 9
Axes, coordinate, 67
Axis, pure imaginary, 243
  real, 243

Base, 10
  of a logarithm, 124
Binomial, 3
  cube of a, 20
  *n*th power of a, 203
  square of a, 17
Binomial formula, 203
  general term of the, 204, 205
  proof of the, 209
Binomial theorem, 202, 203
Braces, 7
Brackets, 7

Characteristic, 128
  rule for the, 130
Clearing of fractions, 55
Coefficients, 2
Coefficients, in terms of the roots, 223
Combinations, 268
  formula for, 269
  total number of, 270
Common difference, 182
Common logarithms, 128
  table of, 302, 303
Common ratio, 188
Commutative law, for addition, 3
  for multiplication, 9
Complete quadratic equation, 141
Completing the square, 143
Complex fractions, 46
Complex plane, 243
Complex number, 119
  absolute value of a, 244
  amplitude of a, 244
  graphical representation of a, 243
  imaginary part of a, 119
  polar form of a, 245
  powers of a, 249
  real part of a, 119
  rectangular form of a, 244
  roots of a, 249
  trigonometric form of a, 245
Complex numbers, conjugate, 119
  fundamental operations with, 120, 242
Conditional equation, 50
Conditional inequality, 197
Consistent equations, 76
Constant, 64
  of proportionality, 97
  of variation, 97
Coordinate, axes, 67
  system, 67
Coordinates of a point, 67
Cube of a binomial, 20
Cubes, factors of the difference of two, 27
  factors of the sum of two, 27
  table of, 301

Degree, of an equation, 212
  of a polynomial, 24
De Moivre's theorem, 249

Denominator, lowest common, 42
Dependent equations, 76
Dependent events, 276
Dependent variable, 65
Depressed equation, 228, 229
Descartes' rule of signs, 225, 226
Determinant, element of a, 284
  expansion of a, 88, 284, 288
  minor of a, 289
  of *n*th order, 283, 284
  of second order, 88
  of third order, 88
  properties of a, 284
  value of a, 284
Difference, common, 182
  of two cubes, 27
  of two squares, 25
Direct variation, 97
Discriminant, 153
Division, definition of, 12
  of fractions, 39
  of multinomials, 13
  of numbers in polar form, 247
  of radicals, 116
  rule of signs for, 13
  synthetic, 214
Double roots, 220

Element, of a determinant, 284
  of a G.P., 188
  of an A.P., 185
Elimination, by addition or subtraction, 78
  by substitution, 77
Ellipse, 167
Equations, 50
  conditional, 50
  consistent, 76
  containing radicals, 150
  degree of, 212
  dependent, 76
  depressed, 228, 229
  equivalent, 51
  exponential, 138
  fractional, 79
  graphical solution of, 73, 74, 158, 169, 170, 218
  inconsistent, 76
  independent, 76
  in quadratic form, 147
  linear, 52
  members of, 50
Equations, of *n*th degree, 212
  operations on, 51
  polynomial, 212
  quadratic, 141
  rational integral, 212
  roots of, 50
  solution of, 50, 70
  systems involving quadratic, 166
  systems of linear, 73, 297, 299
  transformations of, 234
Events, dependent, 276
  independent, 275
  mutually exclusive, 277
Expansion, of a binomial, 202, 203
  of a determinant, 88, 284, 288
  *r*th term of a binomial, 204, 205
Expectation, mathematical, 278
Exponential equations, 138
Exponents, positive integral, 10, 103
  laws of, 103
  negative integral, 105
  zero, 105
Expression, algebraic, 2
  rational integral, 24
Extraneous roots, 52
Extremes, of a progression, 183, 188
  of a proportion, 95

Factors, 2
  common, 30
  literal, 2
  of difference of two cubes, 27
  of difference of two squares, 25
  of sum of two cubes, 27
  prime, 25
Factor theorem, 213
  converse of the, 213
Factorial *n* ($n!$), 201
Formula, binomial, 203
  proof of the binomial, 209
  quadratic, 146
Fourth proportional, 95
Fractions, addition and subtraction of, 42
  complex, 46
  lowest common denominator of, 42
  multiplication and division of, 39
  partial, 253
  proper, 253
  reduction to lowest terms, 37
  rules of signs for, 36, 37

Functions, 65
  graph of a, 69
  linear, 71
  zeros of a, 70
Functional notation, 65
Fundamental operations, 2
  with complex numbers, 242
Fundamental theorem of algebra, 219

General term of the binomial formula, 204, 205
Geometric progression, 188
  elements of a, 188
  extremes of a, 188
  infinite, 192
  means of a, 188
  *n*th term of a, 188
  sum of a, 188, 193
Graph, of a function, 69
  of a linear function, 71
  of a polynomial, 218
  of a quadratic function, 157
Graphical solution, by successive enlargements, 230
  of polynomial equations, 218
  of quadratic equations, 158
  of systems of linear equations, 74
  of systems of quadratic equations, 170
Grouping symbols, 7
  rules for removing and inserting, 8

Horner's method, 237
Hyperbola, 168

Identity, 50
Imaginary number, 118, 119
  pure, 119
Imaginary roots, 153, 221
Imaginary unit (i), 119
Inconsistent equations, 76
Independent equations, 76
Independent events, 275
Independent variable, 65
Induction, mathematical, 206
Inequality, 195
  absolute, 197
  conditional, 197
  members of an, 195
  properties of an, 195, 196
  sense of an, 195
  symbols of an, 2, 195
Interpolation, 133
Inverse variation, 98
Inversions, 283
Irrational equation, 150
Irrational number, 1
Irrational roots, 153
  by Horner's method, 237
  by successive enlargements, 230

Joint variation, 97

Law, associative, 3, 9
  commutative, 3, 9
  distributive, 9
Laws, of exponents, 103
  of logarithms, 126, 127
  of radicals, 111
Like terms, 3
  combining or collecting, 4
Limits for the roots, 224
Linear equations, 52
  systems of, 72, 297, 299
Logarithms, 124
  base of, 124
  characteristic of, 128
  common, 128
  laws of, 126, 127
  mantissa of, 128
  natural, 128
  rule for the characteristic, 130
  table of, 302, 303
  to different bases, 139
Lowest common denominator, 42
Lowest common multiple, 33
Lowest terms of a fraction, 37

Mantissa, 128
Mathematical expectation, 278
Mathematical induction, 206
Mathematical probability, 280, 281
Maximum value of a quadratic function, 160
Mean proportional, 95
Means, arithmetic, 183
  geometric, 188
  of a proportion, 95
Member, of an equation, 50
  of an inequality, 195
Minimum value of a quadratic function, 159
Minor of a determinant, 289
Monomial, 3
Mortality table, 304

Multinomial, 3
  square of a, 17
Multiple, lowest common, 33
  root, 220
Multiplication, 9
  associative law for, 9
  commutative law for, 9
  distributive law for, 9
  of fractions, 39
  of multinomials, 11
  of radicals, 115
  rule of signs for, 10
Mutually exclusive events, 277

Nature of the roots, 153
Negative exponents, 105
Notation for functions, 65
Number, absolute value of a, 3
  complex, 119
  imaginary, 118
  irrational, 1
  literal, 2
  numerical value of a, 3
  pure imaginary, 119
  rational, 1
  real, 1
  roots of a, 107
Number scale, 1

Operations, fundamental, 2
  on equations, 51
  with complex numbers, 120, 242
  with zero, 14
Order of a radical, 108
Ordinate, 67
Origin, 67

Parabola, 157
Parentheses, 7
Partial fractions, 253
Pascal's triangle, 211
Permutations, 264
  circular, 267
  formula for, 265
  of things not all different, 266
Point, coordinates of a, 67
Polynomial, 24
  degree of a, 24
  graph of a, 218
Powers, 10
  of a binomial, 202
  of a complex number, 249
Powers, of i, 119
  table of, 301
Prime factor, 25
Principal root, 108
Probability, experimental, 280, 281
  in multiple events, 275
  in repeated trials, 279
  in single events, 273
  mathematical, 274
Product of the roots, 154, 223
Product of the sum and difference, 16
Progression, arithmetic, 182
  geometric, 188
  infinite geometric, 192
Proof of the binomial formula, 209
Proper fraction, 253
Properties of a determinant, 284
Proportion, 95
  extremes of a, 95
  means of a, 95
Proportional, fourth, 95
  mean, 95
  third, 95
Proportionality constant, 97
Pure, imaginary axis, 243
  imaginary number, 119
  quadratic equation, 141

Quadrants, 67
Quadratic equation, 141, 166
  complete, 141
  discriminant for a, 153
  formula for a, 146
  graphical solution of a, 158
  nature of the roots of a, 153
  product of the roots of a, 154
  pure, 141
  solution by completing the square, 143
  solution by factoring, 142
  standard form of a, 141
  sum of the roots of a, 154
Quadratic form, 147
Quadratic formula, 146
Quadratic surd roots, 221
  theorem on, 221

Radical equation, 150
Radicals, 108
  addition of, 114
  division of, 116
  equations containing, 150
  index of, 108

Radicals, laws of, 111
  like, 114
  multiplication of, 115
  order of, 108
  simplest form of, 111
Radical sign, 108
Radicand, 108
Ratio, 94
  common, 188
Rational integral equation, 212
Rational integral expression, 24
Rationalizing the denominator, 111
Rational number, 1
Rational roots, 153, 227
Real axis, 243
Real number, 1
Real number system, 1
Rectangular coordinate system, 67
Reduction of a fraction, 37
Remainder theorem, 212
Repeated trials of an event, 279
Root, of a number, 107
  of a complex number, 249
  principal, 108
Roots of an equation, 50
  double, 220
  equal, 153
  extraneous, 52, 150
  imaginary, 153, 221
  limits for the, 224
  multiple, 220
  nature of the, 153
  rational, 153, 227
  real, 153
  simple, 220
  triple, 220
Roots, a table of, 301
Rounding off, 133
Rule of signs, 3, 4, 5, 10, 13
  Descartes', 225, 226
  for fractions, 36, 37

Sequence, 182
Simplest form of radicals, 111
Square, of a binomial, 17
  of a multinomial, 17
Squares, a table, 301
  difference of two, 25
Standard form, 141
Stated problems, 57, 85, 162
Statistical probability, 281
Subtraction, definition of, 4
Subtraction, rule of signs for, 5
Successive enlargements, 230
Symbol, factorial, 201
Symbols, of grouping, 7
  of inequalities, 2, 195
Synthetic division, 214
System of coordinates, 67
Systems involving quadratic equations, 166
Systems of linear equations, 73, 299

Table, of logarithms, 302, 303
  of mortality, 304
  of powers and roots, 301
  of trigonometric functions, 305
Terms, 2
  like, 3
  of a determinant, 284
  of an expression, 2
  of a sequence, 182
Theorem, binomial, 202, 203
  De Moivre's, 249
  factor, 213
  fundamental, 219
  remainder, 212
Third proportional, 95
Transformations of equations, 234
Transposing terms, 53
  of an inequality, 196
Trial divisor, 238
Trinomial, 3
  factoring a quadratic, 28
  square of a, 17

Unit, imaginary, 119
Unknowns, 50

Value, absolute, 3, 244
  maximum, 160
  minimum, 159
  numerical, 3
  of a determinant, 284
Variable, 64
  dependent, 65
  independent, 65
Variation, direct, 97
  inverse, 98
  joint, 97

Zero, as an exponent, 105
  of a function, 70
  operations with, 14